Generalized Wilf Conjecture

(Joint work with Dr. Mahir Bilen Can)

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1. Introduction
2. Previous Generalization
3. Unipotent Wilf Conjecture

History (The Wilf Conjecture)

In 1978, Wilf proposed the following conjecture [4]:

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Let S be a complement finite submonoid of $\mathbb{N}_0$, (a.k.a numerical semigroup).

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- *The conductor of S, denoted by $c(S)$ is the smallest integer c such that $c + \mathbb{N}_0 \subseteq S$.*
- *The sporadic elements of S, are elements in S that are less than c. We denote their cardinality by $n(S)$.*
- *The embedding dimension, $e(S)$ of S, is the cardinality of the minimal generating set of $S \backslash 0$.*

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The Wilf conjecture states that for any numerical semigroup S, we have

$$c(S) \leq e(S)n(S)$$

Example

- Let $S = \{0, 3, 6, 7, \ldots\}$. We have $c(S) = 5$, $n(S) = 2$ and $e(S) = |\{3, 7, 8\}| = 3$.

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- Let $S = \{0, 3, 5, 6, 8, \ldots\}$. We have $c(S) = 7$, $n(S) = 4$ and $e(S) = 2$.

$$c(S) = 7 \leq e(S)n(S) = 12$$

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- Let $S = \{0, 3, 6, 7, \dots\}$. We have $c(S) = 5$, $n(S) = 2$ and $e(S) = |\{3, 7, 8\}| = 3$.

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- Let $S = \{0, 2, 5, \dots\}$. We have $c(S) = 4$, $n(S) = 2$ and $e(S) = 2$.

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Example

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Remark: Let $m(S)$ denote the smallest positive integer in $S \backslash 0$, then $e(S) \leq m(S)$.

Connections in Modern Mathematics

Theorem (Weierstrass Gap Theorem)

Let X be a smooth projective curve of genus g and let $P \in X$. Then there exists exactly g numbers $n_1 < \cdots < n_g$ such that $(f)_\infty \neq n_r P$ for any $f \in k(X)$ where $r \in [g]$.

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Example: Let $X = \{[x;y;z] \in \mathbb{P}^2 : xy - z^2 = 0\}$. X is a smooth projective variety. Let $P = [1;1;1] \in X$. For any $f = \frac{g}{h} \in k(X)$, $(f)_\infty = \operatorname{ord}_P(h)$, where

$$\operatorname{ord}_P(h) := \max\{k : h \in \mathfrak{m}_P^k,\ h \notin \mathfrak{m}_P^{k+1}\}$$

For the point P, we have $\mathfrak{m}_P = (x-y, y-z, x-z)$. As $(\frac{x+y}{x-y})_\infty = 1$, so there are no positive integer n for which $(f)_\infty \neq n$. With Macaulay2, one can see that genus of the curve is 0.

Connections in Modern Mathematics

Theorem (Algebraic Codes (M. Homma, S.J Kim , (2001) [3]))

Let $\mathcal{P}$ *be* $\mathbb{F}_q$*-code constructed from rational points of a smooth projective curve* X*. Let* Q_1 *and* Q_2 *be two distinct rational points, each not belonging to* $\mathcal{P}$*. Let* $(\alpha_1, \alpha_2) \in G(Q_1, Q_2)$ *(gap set of* (Q_1, Q_2)*), where* $\alpha_1 \geq 1$ *and*

$$\ell(\alpha_1 Q_1 + \alpha_2 Q_2) = \ell((\alpha_1 - 1)Q_2 + \alpha_2 Q_2)$$

Assume that for some β_1, β_2 *we have* $(\beta_1, \beta_2 - t - 1) \in G(Q_1, Q_2)$ *for all* t *such that*

$$0 \leq t \leq \min\{\beta_2 - 1, 2g - 1 - (\alpha_1 + \alpha_2)\}$$

Put $D = (\alpha_1 + \beta_1 - 1)Q_1 + (\alpha_2 + \beta_2 - 1)Q_2)$ *then the minimum distance of the code* $(X, \mathcal{P}, D)_\Omega$ *satisfies*

$$d \geq \deg G - (2g - 2) + 1$$

Definition

Let $S \subseteq \mathbb{N}_0^d$ be a complement finite submonoid (a.k.a a generalized numerical semigroup). Let $\leq$ be a partial order on $\mathbb{N}_0^d$ such that for $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d) \in \mathbb{N}_0^d$, $x \leq y$ if and only if $x_i \leq y_i$ for all $i = 1, \ldots, r$. Let $H(S) = \mathbb{N}_0^d \backslash S$.

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- The conductor of S, denoted by $c(S)$ is the cardinality of the set

$$\{x \in \mathbb{N}_0^d \,:\, x \leq h \text{ for some } h \in H(S)\}$$

- Let $n(S)$ denote the cardinlity of the set

$$\{x \in S \,:\, x \leq h \text{ for some } h \in H(S)\}$$

- Let $e(S)$ denote the cardinality of the minimal set of generators of S.

Conjecture (Generalized Wilf Conjecture (C. Cisto, M. DiPasquale, G. Failla, Z. Flores, C. Peterson, (2020) [2]))

Let $S \subseteq \mathbb{N}_0^d$ be generalized numerical semigroup. Then

$$dc(S) \leq e(S)n(S)$$

Definition

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Let $\mathbf{U}(n,\mathbb{N})_k := \{(x_{ij})_{1\leq i,j\leq n} \in \mathbf{U}(n,\mathbb{N}) \;:\; k \leq \max_{1\leq i,j\leq n} x_{ij}\} \cup \{\mathbf{1}_n\}$.

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Define

$$\mathbf{P}(n,\mathbb{N}) := \left\{ \begin{pmatrix} 1 & a_1 & a_2 & \dots & a_{n-1} \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} : \{a_1,\dots,a_{n-1}\} \subseteq \mathbb{N}_0 \right\}$$

Define $\mathbf{P}(n,\mathbb{N})_k$ similarly.

Definition

Let $G = \mathbf{U}(n, \mathbb{C})$ and $M = G_{\mathbb{N}}$. Define S to be complement finite submonoid in M. The generating number of S relative to M, denoted by $\mathtt{r}_M(S)$ is the smallest positive integer k such that $\mathbf{U}(n, \mathbb{N})_k \cap S \subseteq S$.

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- $\mathtt{d}_M := \dim G$
- $\mathtt{c}(S) := \mathtt{r}(S)^{\mathtt{d}_M}$. *(Conductor of S relative to M.)*
- $\mathtt{n}(S) := |S \backslash \mathbf{U}(n, \mathbb{N})_{\mathtt{r}_M(S)}| + 1$.
- $\mathtt{e}(S) := \min\{|\mathcal{G}| : \mathcal{G} \text{ generates } S\backslash\{\mathbf{1}_n\}\}$. *(Embedding dimension of S)*
- $\mathtt{g}(S) := |M\backslash S|$. *(Genus of S relative to M.)*

Conjecture (Unipotent Wilf Conjecture)

Let G be a unipotent linear algebraic group. Let $M = G_{\mathbb{N}}$. If S is a complement finite submonoid of M, then we have

$$\mathrm{d}_M(S)\mathrm{c}_M(S) \leq \mathrm{e}(S)\mathrm{n}_M(S).$$

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$$\mathrm{d}_M(S)\mathrm{c}_M(S) \leq \mathrm{e}(S)\mathrm{n}_M(S).$$

$$\binom{n}{2}\mathrm{r}_M(S)^{\binom{n}{2}} \leq \mathrm{e}(S)\mathrm{n}_M(S).$$

Conjecture (Generalized Wilf Conjecture)

Let $M = \mathbf{P}(n, \mathbb{N})$*. If* S *is a complement finite submonoid of* M*, then we have*

$$(n-1)\,\mathrm{r}_M(S)^{n-1} \leq \mathrm{e}(S)\mathrm{n}_M(S).$$

Let ϕ be the monoid isomorphism defined by

$$\phi : \mathbf{P}(n, \mathbb{N}) \longrightarrow \mathbb{N}^{n-1}$$

$$(a_{1j})_{2\leq j\leq n} \longmapsto (a_{12}, a_{13}, \ldots, a_{1n})$$

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$$c(T) := |\{x \in \mathbb{N}_0^d \,:\, x \leq h,\, h \in H(T)\}|, \quad n(T) = |\{x \in T \,:\, x \leq h,\, h \in H(T)\}|$$

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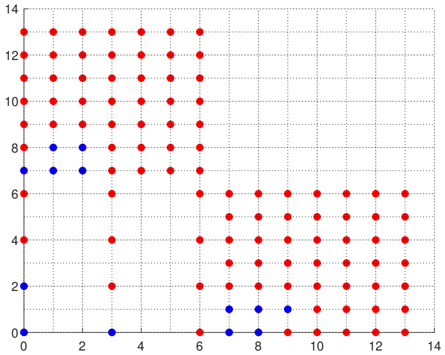
We now have our generalization of [2]. Note that $c(T) \le \mathrm{c}_M(S)$ and $n(T) \le \mathrm{n}_M(S)$.

$$(n-1)c(T) \le \mathrm{c}_M(S) \overset{?}{\le} e(T)n(T) \le \mathrm{e}_M(S)\mathrm{n}_M(S)$$

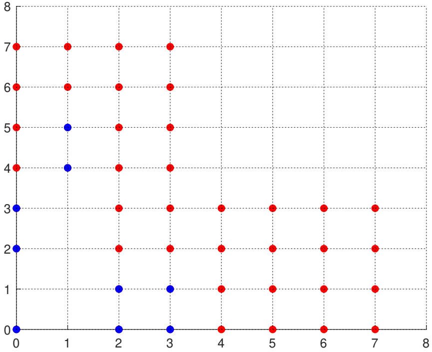
Example ($\mathtt{c}(S) < \mathtt{e}(\phi(S))\mathtt{n}(\phi(S))$)

Let $S = \langle [0,0],[3,0],[0,2],\mathtt{P}_7 \rangle \subseteq \mathbf{P}(3,\mathbb{N})$. Then $\mathtt{c}(S) = 7^2 = 49$, $\mathtt{e}(\phi(S)) = \mathtt{e}(S) = 12$ and $\mathtt{n}(\phi(S)) = 11$. So, $98 \leq 132$. *Note that here* $\mathtt{n}(S) = 12$.



Example ($c(S) > e(\phi(S))n(\phi(S))$)

Let $S = \langle [0,0],[2,0],[3,0],[2,1],[3,1],[0,2],[0,3],\mathtt{P}_4\rangle \subseteq \mathbf{P}(3,\mathbb{N})$. Then $c(S) = 16$, $e(\phi(S)) = 8$ and $n(\phi(S)) = 3$. So, $32 \geq 24$.



Conjecture

Let S be a complement finite submonoid of $M = \mathbf{P}(n, \mathbb{N})$. Then we have

$$\frac{\mathrm{n}_M(S)}{n(\phi(S))} \geq \frac{\mathrm{c}_M(S)}{c(\phi(S))}$$

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In example 1, we had $\mathtt{n}_M(S) = 12$, and $c(\phi(S)) = 48$. So,

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$$\frac{12}{11} \geq \frac{49}{48}$$

In example 2, we had $\mathtt{n}_M(S) = 11$, and $c(\phi(S)) = 8$. So,

$$\frac{11}{3} \geq \frac{16}{8}$$

Theorem (Finite Generation)

Let G be a unipotent linear algebraic group. If S is a unipotent numerical monoid in $M := G_{\mathbb{N}}$, then S is finitely generated. Furthermore, S possesses a unique minimal set of generators.

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Let G be a unipotent linear algebraic group. If S is a unipotent numerical monoid in $M := G_{\mathbb{N}}$, then S is finitely generated. Furthermore, S possesses a unique minimal set of generators.

Proof: One way is to take a generating set and remove the dependent elements. Another approach applies to general unipotent linear algebraic group and use techniques in Borel, H. Chandra (1961) [1].

Theorem

Let S be a unipotent numerical monoid in $M = G_{\mathbb{N}}$, where G is a unipotent linear algebraic group. Then the following inequality holds

$$\left\lfloor \frac{\mathrm{r}_M(S)}{2} \right\rfloor \leq \mathrm{g}_M(S) \leq \mathrm{c}_M(S)$$

Coordinate Monoids

Definition

Let S be a submonoid of $M \in \{\mathbf{U}(n, \mathbb{N}), \mathbf{P}(n, \mathbb{N})\}$. The $(i, j)_{\mathrm{th}}$ $(j > i)$, submonoid of S, denoted by S_{ij}, is the monoid defined by

$$S_{ij} := S \cap \{E_{ij}^{s} \ : \ s \in \mathbb{N}_0\}$$

where E_{ij} is a unipotent matrix with 1 in the $(i, j)^{\mathrm{th}}$ entry and 0 elsewhere

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$$E_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbf{U}(3, \mathbb{N})$$

Restricting to complement finite submonoid $S \subseteq M = \mathbf{P}(n, \mathbb{N})$, we define

$$S_j := S \cap (\{0\} \times \cdots \times \mathbb{N} \times \cdots \times \{0\})$$

We call S_j the basic invariants of a submonoid of M.

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FACTS:

$$\mathrm{c}(S_1) \cdots \mathrm{c}(S_{n-1}) \leq \mathrm{c}_M(S)$$

$$\mathrm{g}(S_1) + \cdots + \mathrm{g}(S_{n-1}) \leq \mathrm{g}_M(S)$$

$$\mathrm{n}(S_1) \cdots \mathrm{n}(S_{n-1}) \leq \mathrm{n}_M(S)$$

Theorem (Bound on the genus of S)

Let $S \subseteq M = \mathbf{P}(n, \mathbb{N})$ be a complement finite submonoid and let $\mathrm{g}_1, \ldots, \mathrm{g}_{n-1}$ denote the genera of $S_1, \ldots, S_{n-1}$ respectively. Let $k = \mathrm{r}_M(S)$. Then we have

$$\mathfrak{e}(\mathrm{g}_1, \ldots, \mathrm{g}_{n-1}) \leq \mathrm{g}_M(S) \leq \sum_{j=1}^{n-1} (-1)^{j-1} k^{n-1-j} \mathfrak{e}_j(\mathrm{g}_1, \ldots, \mathrm{g}_{n-1})$$

where

$$\mathfrak{e}_j(x_1, \ldots, x_{n-1}) = \sum_{1 \leq i_1 < \cdots < i_j \leq n-1} x_{i_1} \cdots x_{i_j}$$

Definition (Thick & Thin Unipotent Numerical Groups)

Let $S \subseteq M = \mathbf{P}(n, \mathbb{N})$ be a complement finite submonoid and let $\mathtt{g}_1, \ldots, \mathtt{g}_{n-1}$ denote the genera of $S_1, \ldots, S_{n-1}$ respectively. Let $\mathtt{n}_1, \ldots, \mathtt{n}_{n-1}$ denote the sporadicities of $S_1, \ldots, S_{n-1}$ respectively. If

$$\sum_{j=1}^{n-1} \mathtt{g}_j = \mathtt{g}_M(S)$$

holds, then S is called a thick monoid.

Definition (Thick & Thin Unipotent Numerical Groups)

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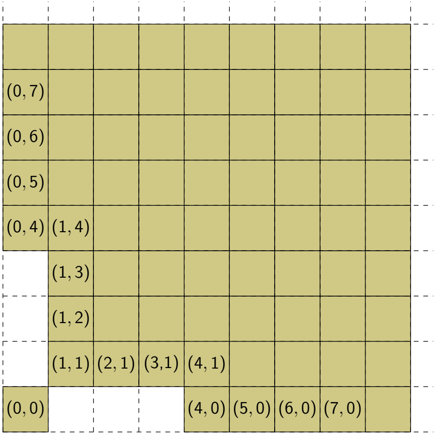

Figure 1: Thick Monoid

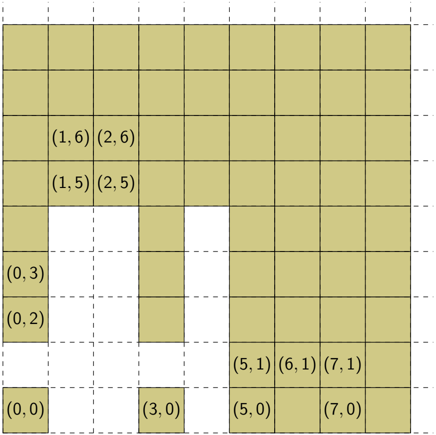

Figure 2: Thin Monoid

Theorem

Let S be a unipotent thick numerical monoid in $M = \mathbf{P}(n, \mathbb{N})$, then the Unipotent Wilf Conjecture holds.

Theorem

Let S be a unipotent thin numerical monoid in $M = \mathbf{P}(n, \mathbb{N})$ with coordinate monoids $S_1, \ldots, S_{n-1}$ and let $\mathtt{e}_1, \ldots, \mathtt{e}_{n-1}$ denote their embedding dimension respectively. Let $\mathtt{m}_i$ denote the smallest non-zero entry in S_i. Then we have the following

$$\mathtt{e}(S) = \sum_{i=1}^{n-1} \mathtt{e}_i + \sum_{I \subseteq \{1,\ldots,n-1\},\, |I| \geq 2} \left(\sum_{i \in I} \mathtt{m}_i \prod_{j \in I \setminus \{i\}} (\mathtt{m}_j - 1) \right)$$

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Theorem

Let S be a unipotent thin numerical monoid in $M = \mathbf{P}(n, \mathbb{N})$ such that $\prod_{j=1}^{n-1} \mathtt{c}_j = \mathtt{r}_M(S)^{n-1}$, then the Unipotent Wilf Conjecture holds which in this case is also equivalent to the Generalized Wilf Conjecture.

[1] Chandra Borel et al. "Arithmetic subgroups of algebraic groups". In: *Bulletin of the American Mathematical Society* 67.6 (1961), pp. 579–583.

[2] Carmelo Cisto et al. "A generalization of Wilfs conjecture for generalized numerical semigroups". In: *Semigroup Forum*. Vol. 101. 2. Springer. 2020, pp. 303–325.

[3] Masaaki Homma and Seon Jeong Kim. "Goppa codes with Weierstrass pairs". In: *Journal of Pure and Applied Algebra* 162.2-3 (2001), pp. 273–290.

[4] Herbert S Wilf. "A circle-of-lights algorithm for the money-changing problem". In: *The American Mathematical Monthly* 85.7 (1978), pp. 562–565.

Thank You

For questions, you can email me at nsakran@tulane.edu