Math 6091/3091: Practice Final ☺

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I am only including material after practice midterm 3.

(1) Are the following matrices diagonalizable? If so, give the diagonal matrix and the change of basis matrix.

(a)

$$\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{pmatrix}$$

Sol:

Characteristic polynomial $= x^3 + 3x^2 + 3x + 1$.
Eigenvalues $= -1$
Eigenvectors $= \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$.
Therefore not diagonalizable.

(b)

$$\begin{pmatrix} 1 & 0 & 1 & -4 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

Sol:

Characteristic polynomial $= (x-1)(x-2)(x-3)(x-4)$.
Eigenvalues $= 1, 2, 3, 4$
Eigenvectors $= \left\{ \lambda = 1\,; \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \lambda = 2\,; \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \lambda = 3\,; \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \lambda = 4\,; \begin{pmatrix} -4/3 \\ 1/2 \\ 0 \\ 1 \end{pmatrix} \right\}$.
Therefore diagonalizable and

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & -4/3 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 1 & -4 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & -4/3 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(2) Find an orthonormal basis for each eigenspace and the spectral decomposition of the following matrix
$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Sol:

Characteristic polynomial $= -x^3 + 6x^2 - 9x + 4$.
Eigenvalues $= 1, 4$
Eigenvectors $= \left\{ \lambda = 1\,;\, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \lambda = 4\,;\, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

(3) Consider the vector space $V = \mathbb{R}[-1,1]$ over $\mathbb{R}$ which consists of continuous functions on the interval $[-1,1]$.

(a) Define $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt$. Show that $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ defines an inner product. **Sol:**

- For $f, g, h \in V$ and $\alpha \in \mathbb{R}$, we have

$$\langle \alpha f + g, h\rangle = \int_{-1}^{1} (\alpha f(t) + g(t))h(t)dt = \alpha \int_{-1}^{1} f(t)h(t)dt + \int_{-1}^{1} g(t)h(t)dt = \alpha\langle f, h\rangle + \langle g, h\rangle.$$

- Clearly $\langle f, g\rangle = \langle g, f\rangle$.
- For any $f \in V$, $\langle f, f\rangle = \int_{-1}^{1} f^2(t)dt \geq 0$ since this is a definite integral of a positive function f^2. Furthermore, the area under the positive curve $f^2(t)$ is zero if and only if $f^2 = 0$.

(b) Consider the subspace $W = \mathrm{Span}\{1, x, x^2\} \subseteq V$. Use Gram-Schmidt process to find an orthonormal basis of W with respect to the given basis. **Sol:**

Let $u_1 = 1$. Note that $\|u_1\| = \sqrt{2}$ Then
$$u_2 = x - \frac{\langle u_1, x\rangle}{\|u_1\|^2} u_1 = x - \frac{\int_{-1}^{1} x dx}{\int_{-1}^{1} dx} = x - 0 = x,$$
with $\|u_2\| = \sqrt{2/3}$. Now
$$u_2 = x^2 - \frac{\langle x^2, x\rangle}{\|x\|^2} x - \frac{\langle x^2, 1\rangle}{\|1\|^2} 1$$
$$= x^2 - \frac{2/3}{2} = x^2 - \frac{1}{3},$$
with $\|u_2\| = \sqrt{8/45}$. Therefore, and orthonormal bases is
$$\left\{ \sqrt{\frac{1}{2}}, \sqrt{\frac{3}{2}} x, \sqrt{\frac{45}{8}} (x^2 - 1/3) \right\}$$

(4) Find all the roots of the following polynomials:

(a) $f(z) = z^3 - z^2 + 2z - 2$. **Sol:**

$z = 1, \sqrt{2}i, -\sqrt{2}i$

(b) $f(z) = z^2 + i$. **Sol:**

$z = \sqrt{2}(-0.5 + i), \sqrt{2}(0.5 - i)$

(5) Find the basis for the kernel and image of the transformation $T : \mathbb{C}^3 \to \mathbb{C}^3$ given by the matrix

$$\begin{pmatrix} 2 & i & 1+i \\ 1-i & 1 & 2 \\ 0 & 1-i & 2 \end{pmatrix}.$$

Sol:

$$\text{Basis for Kernel} = \left\{ \begin{pmatrix} -1 \\ -1-i \\ 1 \end{pmatrix} \right\}.$$

$$\text{Basis for Image} = \left\{ \begin{pmatrix} 2 \\ 1-i \\ 0 \end{pmatrix}, \begin{pmatrix} i \\ 1 \\ 1-i \end{pmatrix} \right\}.$$

(6) Is the set $\{(1-i, 0, i), (1+i, 0, -i), (0, i, 1)\}$ linearly independent over $\mathbb{C}$? **Sol:**

Yes as

$$det \begin{bmatrix} 1-i & 0 & i \\ 1+i & 0 & -i \\ 0 & i & 1 \end{bmatrix} = -2$$

(7) Let $v_1 = (1, i, 2+i)$. Extend this to an orthogonal basis of $\mathbb{C}^3$. **Sol:**

Note that the vector $v_2 = (1, i, 0)$ is orthogonal to the above matrix. Consider the vector $(0, 0, 1)$. So,

$$\begin{aligned} v_3 &= x - \frac{\langle x, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle x, v_2 \rangle}{\|v_2\|^2} v_2 \\ &= (0, 0, 1) - \frac{2+i}{7}(1, i, 2+i) - 0 \\ &= \left(-\frac{2+i}{7}, -\frac{-1+2i}{7}, -\frac{3+4i}{7} \right) \end{aligned}$$

So, an orthogonal basis is $\{v_1, v_2, v_3\}$.

(8) Find the spectral decomposition of the matrix

$$\begin{pmatrix} 1 & 0 & -i \\ 0 & 2 & 0 \\ i & 0 & 2 \end{pmatrix}.$$

Sol:

Characteristic Polynomial$=-x^3 + 5x^2 - 7x + 2$.
Eigenvalues$= 2, \frac{3}{2} \pm \frac{\sqrt{5}}{2}$.

$$\text{Eigenvectors}=\left\{ \lambda = 2; \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \lambda = \tfrac{3+\sqrt{5}}{2}; \begin{pmatrix} 1 \\ 0 \\ \left(\frac{1+\sqrt{5}}{2}\right) i \end{pmatrix}, \lambda = \tfrac{3-\sqrt{5}}{2}; \begin{pmatrix} 1 \\ 0 \\ \left(\frac{1-\sqrt{5}}{2}\right) i \end{pmatrix} \right\}.$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{3+\sqrt{5}}{2} & 0 \\ 0 & 0 & \frac{3-\sqrt{5}}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & \frac{1+\sqrt{5}}{2} i & \frac{1-\sqrt{5}}{2} i \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & -i \\ 0 & 2 & 0 \\ i & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & \frac{1+\sqrt{5}}{2} i & \frac{1-\sqrt{5}}{2} i \end{pmatrix}.$$

(9) Determine whether the subspace $W = \mathrm{Span}\{1+x, x+x^4, x^2-x^5\}$ invariant under the linear transformation

$$T : P_5(\mathbb{R}) \to P_5(\mathbb{R}), \qquad T(p) = p'' - xp' + 3p.$$

Sol:

$$T(1+x) = -x + 3(1+x) = 2x + 3 \notin W.$$

So, not invariant.

(10) Determine whether the subspace $W = \{X \in M_{2\times 2}(\mathbb{R}) \mid \mathrm{Tr}(X) = 0\}$ invariant under the linear transformation

$$T : M_{2\times 2}(\mathbb{R}) \times M_{2\times 2}(\mathbb{R}), \qquad T(X) = AX - XA \qquad \text{where } A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

(*Here $Tr(X)$ denotes trace of X where trace means the sum of diagonal enteries.*)

Sol:
Note that the basis of W is

$$\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

Now,

$$T\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \in W$$

$$T\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \in W$$

$$T\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \in W$$

Therefore, W is stable under T.

(11) Show that if $W \subseteq V$ is invariant under $T : V \to V$ and $\dim W = 1$, then W is spanned by an eigenvector of T.

Sol:
Since $\dim W = 1$, we have $W = \langle x \rangle$ for some $0 \neq x \in V$. Now as W is invariant under T, we have $T(x) \in W$. This means $T(x) = \alpha x$ for some $\alpha \in F$. Therefore, x is an eigenvector of T.

(12) Show that for any linear mapping $T : V \to V$, the subspace $\ker(T)$ and $\mathrm{im}(T)$ are invariant under T.

Sol:
For any $x \in \ker(T)$, we have $T(x) = 0$. Now as $T(0) = 0$, we have $T(x) \in \ker(T)$. Therefore, $\ker(T)$ is invariant under T.

Similarly, let $x \in \mathrm{im}(T)$. Then $T(x) \in \mathrm{im}(T)$. This means $\mathrm{im}(T)$ is invariant under T.

(13) Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 4 & 0 \\ -1 & 0 & 1 \end{pmatrix}$. Find all subspaces of $\mathbb{R}^3$ that are invariant under the linear mapping defined by A.

Sol:

Span of the subsets of $\{v_1, v_2, v_3\}$ where $v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix}$ and $v_3 = \begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix}$. *(These are the eigenvectors).* Finally the following subspace

$$W = \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

is invariant under T.

(14) Find the inverse of the following matrix using the "Cayley-Hamilton method".

$$\begin{pmatrix} 1 & 4 & 2 \\ 2 & 0 & 1 \\ 7 & 1 & 1 \end{pmatrix}.$$

Sol:

Characteristic polynomial$=-x^3 + 2x^2 + 22x + 23$. So,

$$\begin{aligned} -A^3 + 2A^2 + 22A + 23I &= 0 \\ \frac{-1}{23}(-A^2 + 2A + 22I) &= A^{-1} \\ A^{-1} = \frac{-1}{23}&\begin{pmatrix} 1 & 2 & -4 \\ -5 & 13 & -3 \\ -2 & -27 & 8 \end{pmatrix}. \end{aligned}$$

(15) Verify that each of the following mappings is nilpotent, and find the smallest k such that $N^k = 0$.

(a) $N : \mathbb{C}^4 \to \mathbb{C}^4$ defined by

$$\begin{pmatrix} 1 & -3 & 3 & -1 \\ 1 & -3 & 3 & -1 \\ 1 & -3 & 3 & -1 \\ 1 & -3 & 3 & -1 \end{pmatrix}$$

Find the cycle of the element $x = (1, 2, 3, 1)$.

Sol:

$$N = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 1 & -3 & 3 & -1 \\ 1 & -3 & 3 & -1 \\ 1 & -3 & 3 & -1 \end{pmatrix}, \quad N^2 = 0.$$

So, $C(x) = \{(3, 3, 3, 3), (1, 2, 3, 1)\}$

(b) $N : P_4(\mathbb{R}) \to P_4(\mathbb{R})$ defined by

$$N(p) = p'' - 3p'.$$

Find the cycle of the element $p = x^3 - 3x$.

Sol:

$$N(p) = -9x^2 + 6x + 9, \quad N^2(p) = 54x - 36, \quad N^3(p) = -162, \quad N^4(p) = 0.$$

So, $C(x) = \{-162, 54x - 36, -9x^2 + 6x + 9, x^3 - 3x\}$.

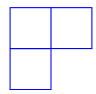
(16) Let $V = \mathbb{R}^3$ and consider the nilpotent map $N : V \to V$ defined by

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \\ 1 & 1 & 1 \end{pmatrix}.$$

Find the canonical form of N and canonical basis of N.

Sol:

It is easy to see that $A^2 = 0$. Now $\dim(\ker(A)) = 2$ and $\dim(\ker(A^2)) = 3$. So, the Tableau is



So, the canonical form is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $x = (1,0,0)$. Then $C(x) = \{(1,-2,1),(1,0,0)\}$. Also for $x = (1,-1,0)$, we have $Ax = 0$. So, $C(x) = \{(1,-1,0)\}$. Therefore, the corresponding canonical basis is

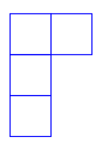
$$\{(1,-2,1),(1,0,0)\} \cup \{(1,-1,0)\}.$$

(17) Find the canonical basis and canonical form for a nilpotent map determined by the following matrix:

$$\begin{pmatrix} 1 & -3 & 3 & -1 \\ 1 & -3 & 3 & -1 \\ 1 & -3 & 3 & -1 \\ 1 & -3 & 3 & -1 \end{pmatrix}$$

Sol:

It is easy to see that $A^2 = 0$. Now $\dim(\ker(A)) = 3$ and $\dim(\ker(A^2)) = 4$. So, the Tableau is



So, the canonical form is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $x = (1,0,0,0)$. Then $C(x) = \{(1,1,1,1),(1,0,0,0)\}$. Also for $x = (0,1,1,0)$ and $x = (3,1,0,0)$, we have $Ax = 0$. Therefore, the corresponding canonical basis is

$$\{(1,1,1,1),(1,0,0,0)\} \cup \{(0,1,1,0)\} \cup \{(3,1,0,0)\}.$$

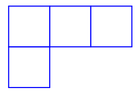
(18) Find the canonical basis and canonical form for a nilpotent map determined by the following matrix:

$$\begin{pmatrix} 4 & 1 & -1 & 2 \\ -4 & -1 & 2 & -1 \\ 4 & 1 & -1 & 2 \\ -4 & -1 & 1 & -2 \end{pmatrix}$$

Sol:

$$A^2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}, \quad A^3 = 0$$

Now $\dim(\ker(A)) = 2, \dim(\ker(A^2)) = 3$ and $\dim(\ker(A^3)) = 4$. So, the Tableau is



So, the canonical form is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The two eigenvectors are

$$\ker(A) = \{(1,-4,0,0),(1,-1,1,-1)\}$$

After some calculation, we can see that the equation $Ax = (1,-4,0,0)$ has no solution. So,

$$C((1,-4,0,0)) = \{(1,-4,0,0)\}.$$

Now, solving the equation $Ax = (1,-1,1,-1)$, we get $x = (-0.25,-1,-1,1)$. Furthermore, solving the equation $A^2x' = (1,-1,1,-1)$, we get $x' = (0,0,2,-1)$. So,

$$C(x') = \{A^2x', Ax', x'\} = \{(1,-1,1,-1),(-4,5,-4,4),(0,0,2,-1)\}.$$

Therefore the canonical basis is

$$\{(1,-1,1,-1),(-4,5,-4,4),(0,0,2,-1)\} \cup \{(1,-4,0,0)\}.$$

Note that nothing changes if we instead picked $x' = (0,0,0,1)$ *or* $x' = (0,0,1,0)$.

(19) Let $T : \mathbb{C}^9 \to \mathbb{C}^9$ be a linear mapping and assume that the characteristic polynomial of T is $((2-i)-x)^3(3-x)^6$. Assume that

$$\dim(\text{Ker}(T-(2-i)I)) = 2$$
$$\dim(\text{Ker}((T-(2-i)I)^2)) = 3$$

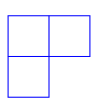
and

$$\dim(\text{Ker}(T-3I)) = 3$$
$$\dim(\text{Ker}(T-3I)^2) = 5$$
$$\dim(\text{Ker}(T-3I)^3) = 6$$

What is the Jordan Canonical Form of T.

Sol:

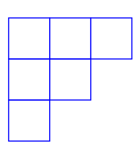
Corresponding to $\lambda = 2-i$, we have



So, the Jordan Block is

$$J_1 \begin{pmatrix} 2-i & 1 & 0 \\ 0 & 2-i & 0 \\ 0 & 0 & 2-i \end{pmatrix}$$

Corresponding to $\lambda = 3$, we have



So, the Jordan Block is

$$J_2 = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

So, the Jordan form is

$$\begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}.$$

(20) Let $T : \mathbb{C}^8 \to \mathbb{C}^8$ be a linear mapping and assume that the characteristic polynomial of T is $(i - x)^4(5 - x)^4$. Assume that

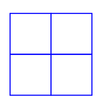
$$\dim(\mathrm{Ker}(T - iI)) = 2$$
$$\dim(\mathrm{Ker}((T - iI)^2)) = 4$$

and

$$\dim(\mathrm{Ker}(T - 3I)) = 2$$
$$\dim(\mathrm{Ker}(T - 3I)^2) = 3$$
$$\dim(\mathrm{Ker}(T - 3I)^3) = 4$$

What is the Jordan Canonical Form of T.

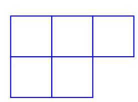
Sol:
Corresponding to $\lambda = i$, we have



So, the Jordan Block is

$$J_1 \begin{pmatrix} i & 1 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & 0 & i \end{pmatrix}$$

Corresponding to $\lambda = 3$, we have



So, the Jordan Block is

$$J_2 = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

So, the Jordan form is

$$\begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}.$$

(21) Find the Jordan Canonical form and Jordan Canonical basis for each of the following mappings:

(a) $T : \mathbb{C}^4 \to \mathbb{C}^4$ defined by the matrix

$$\begin{pmatrix} 1 & -3 & 5 \\ -2 & -6 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Sol:

Characteristic Polynomial$=-x^3 - 4x^2 + 17x - 12$.
Eigenvalues $= 1, -\frac{5}{2} \pm \frac{\sqrt{73}}{2}$.
The Jordan Canonical basis consists of eigenvectors. And the Jornal canonical Form is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{-5+\sqrt{73}}{2} & 0 \\ 0 & 0 & \frac{-5-\sqrt{73}}{2} \end{pmatrix}$$

(b) $T : \mathbb{C}^4 \to \mathbb{C}^4$ defined by the matrix

$$\begin{pmatrix} 1 & -3 & 5 & 3 \\ -2 & -6 & 0 & 13 \\ 0 & 0 & 1 & 0 \\ -1 & -4 & 7 & 8 \end{pmatrix}$$

Sol:

Characteristic Polynomial$=(x-1)^4$.
Consider the nilpotent matrix

$$N = A - I = \begin{pmatrix} 0 & -3 & 5 & 3 \\ -2 & -7 & 0 & 13 \\ 0 & 0 & 0 & 0 \\ -1 & -4 & 7 & 7 \end{pmatrix}$$

So,

$$N^2 = \begin{pmatrix} 3 & 9 & 21 & -18 \\ 1 & 3 & 81 & -6 \\ 0 & 0 & 0 & 0 \\ 1 & 3 & 44 & -6 \end{pmatrix}, \quad N^3 = \begin{pmatrix} 0 & 0 & -111 & 0 \\ 0 & 0 & -37 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -37 & 0 \end{pmatrix}, \quad N^4 = 0.$$

So,

$$\dim(\ker(N)) = 1, \quad \dim(\ker(N^2)) = 2, \quad \dim(\ker(N^3)) = 3, \quad \dim(\ker(N^4)) = 4.$$

So the Tableau is

$$\begin{array}{|c|c|c|c|}\hline \quad & \quad & \quad & \quad \\ \hline \end{array}$$

Therefore the Jordan Canonical Form is

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Picking $x = (1,1,1,1)$ at random, we get

$$C(x) = \{(-111,-37,0,-37),(15,79,0,42),(5,4,0,9),(1,1,1,1)\}.$$

Therefore the canonical basis is

$$\{(-111,-37,0,-37),(15,79,0,42),(5,4,0,9),(1,1,1,1)\}.$$

(c) Let $V = \operatorname{Span}\{\sin(x), \cos(x), e^{4x}, xe^{4x}, x^2e^{4x}\}$, and $T : V \to V$ be a mapping defined by

$$T(f) = f' - 2f.$$

Sol:

Lets write in terms of matrix. Let $E_1 = \sin(x)$, $E_2 = \cos(x)$, $E_3 = e^{4x}$, $E_4 = xe^{4x}$, $E_5 = x^2e^{4x}$. Now

$$\begin{aligned}
T(E_1) &= -2E_1 + E_2 \\
T(E_2) &= -E_1 - 2E_2 \\
T(E_3) &= 2E_3 \\
T(E_4) &= E_3 + 2E_4 \\
T(E_5) &= 2E_4 + 2E_5.
\end{aligned}$$

So, the matrix is

$$A = \begin{pmatrix} -2 & -1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

which gives a transformation $T : \mathbb{C}^5 \to \mathbb{C}^5$. Now the eigen values are $2, 2, 2, -2+i, -2-i$. Considering

$$A - 2I = \begin{pmatrix} -4 & -1 & 0 & 0 & 0 \\ 1 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

we can take $N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$. We have

$$N^2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad N^3 = 0.$$

Now $\dim(\ker(N)) = 1$, $\dim(\ker(N^2)) = 2$, $\dim(\ker(N^3)) = 3$. So, the Tableau is

$$\boxed{}\boxed{}\boxed{}$$

So, the Jordan canonical form is

$$\begin{pmatrix} -2+i & 0 & 0 & 0 & 0 \\ 0 & -2-i & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

The canonical basis of N is $C((1,1,1)) = \{(2,0,0), (1,2,0), (1,1,1)\}$ *(I picked $(1,1,1)$ at random and it worked)*. Therefore the canonical basis is

$$\{\text{Eigenvector of } -2+i\} \cup \{\text{Eigenvector of } -2-i\} \cup \{(0,0,2,0,0), (0,0,1,2,0), (0,0,1,1,1)\}.$$