Math 6091/3091: Practice Midterm 2 Solution

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(1) Prove that if $\{v_1, \cdots, v_n\}$ are linearly independent vectors in the vector space V then the following vectors

$$\{v_1 - v_2, v_2 - v_3, \cdots, v_{n-1} - v_n, v_n\}$$

are also linearly independent.

Sol:
Suppose the vectors $\{v_1 - v_2, v_2 - v_3, \cdots, v_{n-1} - v_n, v_n\}$ were not linearly independent. So, there exists $\alpha_1, \cdots, \alpha_n$, not all zero such that

$$\alpha_1(v_1 - v_2) + \alpha_2(v_2 - v_3) + \cdots + \alpha_{n-1}(v_{n-1} - v_n) + \alpha_n v_n = 0.$$

But then

$$\begin{aligned} \alpha_1(v_1 - v_2) + \alpha_2(v_2 - v_3) + \cdots + \alpha_{n-1}(v_{n-1} - v_n) + \alpha_n v_n &= 0 \\ \alpha_1 v_1 + (\alpha_2 - \alpha_1)v_2 + (\alpha_3 - \alpha_2)v_3 + \cdots + (\alpha_n - \alpha_{n-1})v_n &= 0 \end{aligned}$$

But then this implies $\alpha_1 = 0$ and $\alpha_n = \alpha_{n-1} = \cdots = \alpha_2 = \alpha_1 = 0$, a contradiction. So, the vectors $\{v_1 - v_2, v_2 - v_3, \cdots, v_{n-1} - v_n, v_n\}$ must be linearly independent.

(2) Let V and W be finite-dimensional vector spaces over the field $\mathbf{F}$. Show that $\dim V = \dim W$ if and only if V is isomorphic to W (i.e. you can construct a bijective linear map between V and W.)

Sol:
Let $\dim V = \dim W$. Suppose $\{v_1, \cdots, v_n\}$ and $\{w_1, \cdots, w_n\}$ be basis for V and W respectively, having the same number of elements as $\dim V = \dim W$. Then the map $T : V \longrightarrow W$ such as $T(v_i) = T(w_i)$ for all $i = 1, \cdots, n$ is a bijective linear map. Thus, V is isomorphic to W.

Conversely, let V is isomorphic to W. Then we have a bijective linear map T from V to W. Then there exists basis $\{v_1, \cdots, v_n\}$ of V and $\{w_1, \cdots, w_m\}$ of W such that $T(v_i) = w_i$ for all i. But then since T is a bijective linear transformation, we must have $n = m$. So, $\dim V = \dim W$.

(3) Determine whether $T : M_{22} \longrightarrow \mathbb{R}$ such that

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 3a - 4b + c - d$$

is a linear transformation or not.

Sol:
Let $\begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}, \begin{bmatrix} w_1 & w_2 \\ w_3 & w_4 \end{bmatrix} \in M_{22}$ and $\alpha, \beta \in \mathbb{R}$, then

$$\begin{aligned} T\left(\alpha \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix} + \beta \begin{bmatrix} w_1 & w_2 \\ w_3 & w_4 \end{bmatrix}\right) &= T\left(\begin{bmatrix} \alpha_1 v_1 + \beta w_1 & \alpha v_2 + \beta w_2 \\ \alpha v_3 + \beta w_3 & \alpha v_4 + \beta w_4 \end{bmatrix}\right) \\ &= 3(\alpha v_1 + \beta w_1) - 4(\alpha v_2 - \beta w_2) + (\alpha v_3 + \beta w_3) - (\alpha v_4 + \beta w_4) \\ &= \alpha(3v_1 - 4v_2 + v_3 - v_4) + \beta(3w_1 - 4w_2 + w_3 - w_4) \\ &= \alpha T\left(\begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}\right) + \beta T\left(\begin{bmatrix} w_1 & w_2 \\ w_3 & w_4 \end{bmatrix}\right) \end{aligned}$$

(4) Consider the basis $S = \{(1,1,1),(1,1,0),(1,0,0)\}$ for $\mathbb{R}^3$. Consider the linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ given by

$$T(1,1,1) = (1,0), \qquad T(1,1,0) = (2,-1), \qquad T(1,0,0) = (4,3).$$

Find a formula for $T(x_1,x_2,x_3)$ and then use that formula to compute $T(2,-3,5)$.

Sol:

For any $(x_1,x_2,x_3) \in \mathbb{R}^3$. Let $c_1,c_2,c_3 \in \mathbb{R}$ such that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1+c_2+c_3 \\ c_1+c_2 \\ c_1 \end{bmatrix}$$

which implies $c_1 = x_3$, $c_2 = x_2 - x_3$ and $c_3 = x_1 - x_2$. So,

$$\begin{aligned} T(x_1,x_2,x_3) &= T(c_1(1,1,1) + c_2(1,1,0) + c_3(1,0,0)) \\ &= c_1T(1,1,1) + c_2T(1,1,0) + c_3T(1,0,0) \\ &= x_3(1,0) + (x_2-x_3)(2,-1) + (x_1-x_2)(4,3) \\ &= (4x_1 - 2x_2 - x_3, 3x_1 - 4x_2 + x_3). \end{aligned}$$

Now, $T(2,-3,5) = (9,23)$.

(5) Consider the basis $S = \{(-2,1),(1,3)\}$ for $\mathbb{R}^2$. Consider the linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by

$$T(-2,1) = (-1,2,0), \qquad T(1,3) = (0,-3,5).$$

Find a formula for $T(x_1,x_2)$ and then use that formula to compute $T(2,-3)$.

Sol:

$$\begin{aligned} (x_1,x_2) &= c_1(-2,1) + c_2(1,3) \\ &= (-2c_1 + c_2, c_1 + 3c_2) \end{aligned}$$

We get $c_1 = \frac{-3}{7}x_1 + \frac{1}{7}x_2$ and $c_2 = \frac{1}{7}x_1 + \frac{2}{7}x_2$. So,

$$\begin{aligned} T(x_1,x_2) &= c_1(-1,2,0) + c_2(0,-3,5) \\ &= \left(\frac{-3}{7}x_1 + \frac{1}{7}x_2\right)(-1,2,0) + \left(\frac{1}{7}x_1 + \frac{2}{7}x_2\right)(0,-3,5) \\ &= \left(\frac{3}{7}x_1, -\frac{9}{7}x_1 - \frac{4}{7}x_2, \frac{5}{7}x_1 + \frac{10}{7}x_2\right) \end{aligned}$$

So, $T(2,-3) = \left(\frac{6}{7}, -\frac{6}{7}, \frac{-20}{7}\right)$

(6) Consider the basis $S = \{(1,2,1),(2,9,0),(3,3,4)\}$ for $\mathbb{R}^3$. Consider the linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ given by

$$T(1,2,1) = (1,0), \qquad T(2,9,0) = (-1,1), \qquad T(3,3,4) = (0,1).$$

Find a formula for $T(x_1,x_2,x_3)$ and then use that formula to compute $T(7,13,7)$.

Sol:

We get $c_1 = -36x_1 + 8x_2 + 21x_3$, $c_2 = 5x_1 - x_2 - 3x_3$, $c_3 = 9x_1 - 2x_2 - 5x_3$. Therefore,

$$T(x_1,x_2,x_3) = (-41x_1 + 9x_2 + 24x_3, 14x_1 - 3x_2 - 8x_3)$$

True-False Exercises

TF. In parts (a)–(i) determine whether the statement is true or false, and justify your answer.

(a) If $T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)$ for all vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in V and all scalars c_1 and c_2, then T is a linear transformation.

(b) If $\mathbf{v}$ is a nonzero vector in V, then there is exactly one linear transformation $T: V \to W$ such that $T(-\mathbf{v}) = -T(\mathbf{v})$.

(c) There is exactly one linear transformation $T: V \to W$ for which $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u} - \mathbf{v})$ for all vectors $\mathbf{u}$ and $\mathbf{v}$ in V.

(d) If $\mathbf{v}_0$ is a nonzero vector in V, then the formula $T(\mathbf{v}) = \mathbf{v}_0 + \mathbf{v}$ defines a linear operator on V.

(e) The kernel of a linear transformation is a vector space.

(f) The range of a linear transformation is a vector space.

(g) If $T: P_6 \to M_{22}$ is a linear transformation, then the nullity of T is 3.

(h) The function $T: M_{22} \to R$ defined by $T(A) = \det A$ is a linear transformation.

(i) The linear transformation $T: M_{22} \to M_{22}$ defined by

$$T(A) = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} A$$

has rank 1.

TRUE,FALSE,TRUE,FALSE,TRUE,TRUE,FALSE,FALSE,TRUE

(7) Let $T : \mathbf{P}_2 \longrightarrow \mathbf{P}_3$ given by $T(p(x)) = xp(x)$. Find the Kernel of T. Is T onto?

Sol:
$Ker = 0$ and T is not onto.

(8) Find the kernel of the transformation T given by the following matrix

$$\begin{bmatrix} 2 & 3 & 4 & 1 & 1 \\ 1 & 1 & 8 & 8 & 8 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix}.$$

Sol:

$$\begin{bmatrix} 2 & 3 & 4 & 1 & 1 \\ 1 & 1 & 8 & 8 & 8 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 20 & 0 & 0 \\ 0 & 1 & -12 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

So, the kernel is

$$span \left\{ \begin{bmatrix} -20 \\ 12 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

(9) Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^1$ given by $T(x_1, x_2, x_3) = 3x_3 + 2x_2 - x_1$. Find the preimage of 5.

Sol:
The Kernel of the transformation is

$$Ker = span\{(3, 0, 1), (2, 1, 0)\}.$$

Note that $T(-5, 0, 0) = 5$. So, the preimage of 5 is the set

$$\left\{ \begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix} + t_1 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \mid t_1, t_2 \in \mathbb{R} \right\}.$$

(10) Consider the linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ given by

$$T(x_1, x_2, x_3, x_4, x_5) = (x_4 + x_5, 2x_2 + 3x_3 - x_5, -x_1 + 2x_2 + 3x_3).$$

Find the preimage of $(0, 5, 1)$.

Sol:
The matrix of the map is

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 3 & 0 & -1 \\ -1 & 2 & 3 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1.5 & 0 & -0.5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

So, the Kernel of T is

$$span \left\{ \begin{bmatrix} 1 \\ 0.5 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1.5 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

We get the particular solution by

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 3 & 0 & -1 & 5 \\ -1 & 2 & 3 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 4 \\ 0 & 1 & 1.5 & 0 & -0.5 & 2.5 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

So, the particular solution is $\begin{bmatrix} 4 \\ 2.5 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ Therefore, the preimage is the set

$$\left\{ \begin{bmatrix} 4 \\ 2.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_1 \begin{bmatrix} 1 \\ 0.5 \\ 0 \\ -1 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ -1.5 \\ 1 \\ 0 \\ 0 \end{bmatrix} \mid t_1, t_2 \in \mathbb{R} \right\}$$

(11) Consider the linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ given by $T(x, y) = (x + 2y, x + 3y, x, y)$. Find the basis for the Kernel and Image of T.

Sol:
The matrix of the linear transformation is

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

So, the Kernel is 0 and the image of the transformation is

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

(12) Find the inverse of the following matrix

$$\begin{bmatrix} 5 & 4 \\ 6 & 7 \end{bmatrix}.$$

Sol:
The inverse is

$$\frac{1}{11} \begin{bmatrix} 7 & -4 \\ -6 & 5 \end{bmatrix}$$

(13) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear transformation given by

$$T(x_1, x_2, x_3) = (3x_1 + x_2, -2x_1 - 4x_2 + 3x_3, 5x_1 + 4x_2 - 2x_3).$$

Determine whether T is invertible. If yes, find $T^{-1}(x_1, x_2, x_3)$ and use it to compute $T^{-1}(1, 2, 3)$

Sol:
The matrix of the transformation is

$$\begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

The inverse of the matrix is

$$\begin{bmatrix} 4 & -2 & -3 \\ -11 & 6 & 9 \\ -12 & 7 & 10 \end{bmatrix}$$

So,

$$T^{-1}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 4 & -2 & -3 \\ -11 & 6 & 9 \\ -12 & 7 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -9 \\ 28 \\ 32 \end{bmatrix}.$$

(14) Let

$$T_1(x, y) = (x + y, y, -x), \quad T_2(x, y, z) = (0, x + y + z, 3y), \quad T_3(x, y, z) = (3x + 2y, 4z - x - 3y).$$

Find the formula for $T_3 \circ T_2 \circ T_1(x, y)$ and use it to compute $T_3 \circ T_2 \circ T_1(10, 10)$.

Sol:

$$A_{T_1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$A_{T_2} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 3 & 0 \end{bmatrix}$$

$$A_{T_3} = \begin{bmatrix} 3 & 2 & 0 \\ -1 & -3 & 4 \end{bmatrix}$$

So, the matrix of $T_3 \circ T_2 \circ T_1$ is

$$A_{T_3 \circ T_2 \circ T_1} = \begin{bmatrix} 3 & 2 & 0 \\ -1 & -3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 4 \\ 0 & 6 \end{bmatrix}$$

So, $T_3 \circ T_2 \circ T_1(x, y) = (4y, 6y)$. Thus, $T_3 \circ T_2 \circ T_1(10, 10) = (40, 60)$.

True-False Exercises

TF. In parts (a)–(f) determine whether the statement is true or false, and justify your answer.

(a) The composition of two linear transformations is also a linear transformation.

(b) If $T_1\colon V \to V$ and $T_2\colon V \to V$ are any two linear operators, then $T_1 \circ T_2 = T_2 \circ T_1$.

(c) The inverse of a one-to-one linear transformation is a linear transformation.

(d) If a linear transformation T has an inverse, then the kernel of T is the zero subspace.

(e) If $T: R^2 \to R^2$ is the orthogonal projection onto the x-axis, then $T^{-1}\colon R^2 \to R^2$ maps each point on the x-axis onto a line that is perpendicular to the x-axis.

(f) If $T_1\colon U \to V$ and $T_2\colon V \to W$ are linear transformations, and if T_1 is not one-to-one, then neither is $T_2 \circ T_1$.

FALSE, FALSE, FALSE,TRUE,FALSE,TRUE

(15) The Fundamental Theorem of Calculus implies that integration and differentiation reverse the actions of each other. Define the transformation $D : \mathbf{P}_n \longrightarrow \mathbf{P}_{n-1}$ by $D(p(x)) = p'(x)$ and define $I : \mathbf{P}_{n-1} \longrightarrow \mathbf{P}_n$ by

$$I(p(x)) = \int_0^x p(t)dt$$

Verify that D and I are linear transformations. Explain why I is not the inverse transformation of D. Find the basis of the image of J.

Sol:
We know from calculus that $D(\alpha f+\beta g) = \alpha D(f)+\beta D(g)$ and $I(\alpha f+\beta g) = \alpha I(f)+\beta I(g)$, therefore, D and I are linear transformations. I cannot be the inverse of D as $I(D(p(x))) = 0$ for any constant polynomial $p(x)$ (e.g. $p(x) = 5$). The basis for the image of I is

$$\{x, x^2, \cdots, x^n\}.$$

(16) Consider the map $T : M_{22} \longrightarrow \mathbb{R}^4$ given by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a, a+b, a+b+c, a+b+c+d).$$

Is T an isomorphism i.e. is T a bijective linear map?

Sol:
Consider the following basis

$$\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

for $M_{2,2}$. Now, since

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = (1,1,1,1), \quad T\left(\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right) = (1,2,2,2), \quad T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = (1,2,3,3), \quad T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = (1,2,3,4)$$

and the vectors

$$\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}\right\}$$

forms a basis of $\mathbb{R}^4$, we get that T is a bijective linear map.

(17) Determine whether the following is true or false:

- There is a subspace of M_{23} that is isomorphic to $\mathbb{R}^4$.
- Isomorphic finite-dimensional vector spaces must have the same number of basis vectors.
- $\mathbb{R}^n$ is isomorphic to $\mathbb{R}^{n+1}$.

Sol:
TRUE,TRUE,FALSE

(18) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be the linear transformation defined by

$$T(x_1, x_2) = (x_2, -5x_1 + 13x_2, -7x_1 + 16x_2)$$

Find the matrix for the linear transformation T with respect to the bases $B = \{u_1, u_2\}$ of $\mathbb{R}^2$ and $B' = \{v_1, v_2, v_3\}$ for $\mathbb{R}^3$, where

$$u_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}; \qquad v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Sol:

We have

$$T(u_1) = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \quad T(u_2) = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}.$$

Now, we write their image in terms of the bases B', we get

$$[T(u_1)]_{B'} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}_{B'} = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \quad [T(u_2)]_{B'} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}_{B'} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$$

Now, the matrix of $[T]_B^{B'}$ is

$$\begin{bmatrix} [T(u_1)]_{B'} & [T(u_2)]_{B'} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}.$$

(19) Let $T : \mathbf{P}_2 \longrightarrow \mathbf{P}_2$ be a linear transformation defined by

$$T(p(x)) = p(3x - 5),$$

i.e. $T(a_0 + a_1x + a_2x^2) = a_0 + a_1(3x-5) + a_2(3x-5)^2$. Find $[T]_B$ relative to the basis $B = \{1, x, x^2\}$. Use it to compute $T(1 + 2x + 3x^2)$.

Sol:

We have

$$T(1) = 1, \quad T(x) = -5 + 3x, \quad T(x^2) = 25 - 30x + 9x^2.$$

Now, we write their image in terms of the bases B, we get

$$[T(1)]_{B'} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_B, \quad [T(x)]_B = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}_B, \quad [T(x^2)]_B = \begin{bmatrix} 25 \\ -30 \\ 9 \end{bmatrix}_B$$

Now, the matrix of $[T]_B^B$ is

$$\begin{bmatrix} [T(1)]_B & [T(x)]_B & [T(x^2)]_B \end{bmatrix} = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix}.$$

(20) Let $V = \mathbb{R}^3$ and $W = \mathbb{R}^3$, and let $T : V \longrightarrow W$ be a linear transformation. Let

$$\alpha = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ be basis of } V \text{ and } \beta = \left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\} \text{ be basis of } W$$

Suppose T is given by

$$\begin{aligned} T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) &= \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \\ T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right) &= - \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \\ T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) &= \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}. \end{aligned}$$

Find $T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$.

Sol:

We know that

$$\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

So,

$$\begin{aligned} T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) &= \frac{1}{2} \left(\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right) \\ &= \frac{1}{2} \left(\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} 2.5 \\ 1.5 \\ 2 \end{bmatrix} \end{aligned}$$