Math 6091/3091: Practice Midterm 3 Solution

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Do **all** of the following problems.

(1) Find the determinants of the following matrices.

(a)

$$\begin{pmatrix} 3 & 3 & 1 \\ 1 & 0 & -4 \\ 1 & -3 & 5 \end{pmatrix}$$

Sol

$$det = -66$$

(b)

$$\begin{pmatrix} 1 & k & k^2 \\ 1 & k & k^2 \\ 1 & k & k^2 \end{pmatrix}$$

Sol

$$det = 0$$

(c)

$$\begin{pmatrix} 3 & 3 & 0 & 5 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{pmatrix}$$

Sol

$$det = -240$$

(2) Show that the value of the determinant is independent of θ.

(a)

$$\begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}$$

Sol

$$det = \sin^2\theta + \cos^2\theta = 1$$

(b)

$$\begin{pmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta - \cos\theta & \sin\theta + \cos\theta & 1 \end{pmatrix}$$

Sol

$$det = \sin^2\theta + \cos^2\theta = 1$$

TF. In parts (a)–(j) determine whether the statement is true or false, and justify your answer.

(a) The determinant of the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad + bc$.

(b) Two square matrices that have the same determinant must have the same size.

(c) The minor M_{ij} is the same as the cofactor C_{ij} if $i + j$ is even.

(d) If A is a 3×3 symmetric matrix, then $C_{ij} = C_{ji}$ for all i and j.

(e) The number obtained by a cofactor expansion of a matrix A is independent of the row or column chosen for the expansion.

(f) If A is a square matrix whose minors are all zero, then $\det(A) = 0$.

(g) The determinant of a lower triangular matrix is the sum of the entries along the main diagonal.

(h) For every square matrix A and every scalar c, it is true that $\det(cA) = c\det(A)$.

(i) For all square matrices A and B, it is true that

$$\det(A + B) = \det(A) + \det(B)$$

(j) For every 2×2 matrix A it is true that $\det(A^2) = (\det(A))^2$.

False, False, True, True, True, True, False, Falsa, False, True

(3) Let

$$A = \begin{pmatrix} -2 & 8 & 1 & 4 \\ 3 & 2 & 5 & 1 \\ 1 & 10 & 6 & 5 \\ 4 & -6 & 4 & -3 \end{pmatrix}$$

Show that $\det(A) = 0$ without directly evaluating the integral. **Sol**

$$\begin{vmatrix} -2 & 8 & 1 & 4 \\ 3 & 2 & 5 & 1 \\ 1 & 10 & 6 & 5 \\ 4 & -6 & 4 & -3 \end{vmatrix} \xrightarrow{R_1 = R_1 + 2R_3} \begin{vmatrix} 0 & 28 & 13 & 14 \\ 3 & 2 & 5 & 1 \\ 1 & 10 & 6 & 5 \\ 4 & -6 & 4 & -3 \end{vmatrix} \xrightarrow{R_2 = R_2 - 3R_3} \begin{vmatrix} 0 & 28 & 13 & 14 \\ 0 & -28 & -13 & -14 \\ 1 & 10 & 6 & 5 \\ 4 & -6 & 4 & -3 \end{vmatrix} = 0$$

(4) Given that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6.$$

Find

$$\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g+3a & h+3b & i+3c \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ 4g+3a & 4h+3b & 4i+3c \end{vmatrix}$$

Sol

$$\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g+3a & h+3b & i+3c \end{vmatrix} = -12 \quad \text{and} \quad \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ 4g+3a & 4h+3b & 4i+3c \end{vmatrix} = 2 * 4 * (-6) = -48$$

TF. In parts (a)–(f) determine whether the statement is true or false, and justify your answer.

(a) If A is a 4×4 matrix and B is obtained from A by interchanging the first two rows and then interchanging the last two rows, then $\det(B) = \det(A)$.

(b) If A is a 3×3 matrix and B is obtained from A by multiplying the first column by 4 and multiplying the third column by $\frac{3}{4}$, then $\det(B) = 3\det(A)$.

(c) If A is a 3×3 matrix and B is obtained from A by adding 5 times the first row to each of the second and third rows, then $\det(B) = 25\det(A)$.

(d) If A is an $n \times n$ matrix and B is obtained from A by multiplying each row of A by its row number, then

$$\det(B) = \frac{n(n+1)}{2}\det(A)$$

(e) If A is a square matrix with two identical columns, then $\det(A) = 0$.

(f) If the sum of the second and fourth row vectors of a 6×6 matrix A is equal to the last row vector, then $\det(A) = 0$.

True, True, False, True, True, True

(5) Use Cramer's Rule to solve the following system of equations.

$$3x_1 - x_2 + x_3 = 4$$
$$-x_1 + 7x_2 - 2x_3 = 1$$
$$2x_1 + 6x_2 + x_3 = 5$$

Sol

$$x_1 = \frac{58}{40}, \qquad x_2 = \frac{14}{40}, \qquad x_3 = 0.$$

(6) Solve for the value of y without finding the value of x, z and w.

$$4x + y + z + w = 6$$
$$3x + 7y - z + w = 1$$
$$7x + 3y - 5z + 8w = -3$$
$$x + y + z + 2w = 3$$

Sol

$$y = 0.$$

(7) Prove that a square matrix A is invertible if and only if $A^T A$ is invertible.

Sol

Let A be invertible then $A^{-1}(A^{-1})^T$ is well defined and is the inverse of $A^T A$. Conversely, suppose $A^T A$ is invertible. Then $(A^T A)^{-1} = A^{-1}(A^{-1})^T$ implies A^{-1} exists. So, A is invertible.

(8) Prove that if A is a square matrix then $\det(A^T A) = \det(AA^T)$. **Sol**

$$det(AA^T) = det(A)det(A^T) = det(A^T)det(A) = det(A^T A)$$

(9) Prove that if $\det(A) = 1$ and all the entries of A are integers, then all the entries of A^{-1} are integers.
Sol

Since, $A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = \text{adj}(A)$. Now as entries in the adjoint of A are just addition and multiplication operation on integers. So, the entries of the adjoint are integers. Thus, the entries of A^{-1} are integers.

TF. In parts (a)–(l) determine whether the statement is true or false, and justify your answer.

(a) If A is a 3×3 matrix, then $\det(2A) = 2\det(A)$.

(b) If A and B are square matrices of the same size such that $\det(A) = \det(B)$, then $\det(A + B) = 2\det(A)$.

(c) If A and B are square matrices of the same size and A is invertible, then

$$\det(A^{-1}BA) = \det(B)$$

(d) A square matrix A is invertible if and only if $\det(A) = 0$.

(e) The matrix of cofactors of A is precisely $[\operatorname{adj}(A)]^T$.

(f) For every $n \times n$ matrix A, we have

$$A \cdot \operatorname{adj}(A) = (\det(A))I_n$$

(g) If A is a square matrix and the linear system $A\mathbf{x} = \mathbf{0}$ has multiple solutions for $\mathbf{x}$, then $\det(A) = 0$.

(h) If A is an $n \times n$ matrix and there exists an $n \times 1$ matrix $\mathbf{b}$ such that the linear system $A\mathbf{x} = \mathbf{b}$ has no solutions, then the reduced row echelon form of A cannot be I_n.

(i) If E is an elementary matrix, then $E\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(j) If A is an invertible matrix, then the linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if the linear system $A^{-1}\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(k) If A is invertible, then $\operatorname{adj}(A)$ must also be invertible.

(l) If A has a row of zeros, then so does $\operatorname{adj}(A)$.

(10) Solve for x

$$\begin{vmatrix} x & -1 \\ 3 & 1-x \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ 2 & x & -6 \\ 1 & 3 & x-5 \end{vmatrix}$$

Sol

$$\begin{aligned} &x(1-x)+3 = x(-6-2x+10) - 3(-6+6) \\ &-x^2+x+3 = x^2-2x \\ &2x^2-3x-3=0 \\ &\qquad x = \frac{3}{4} \pm \frac{\sqrt{33}}{4} \end{aligned}$$

(11) Find the characteristic equation, the eigenvalues, and bases for the eigenspaces of the matrix.

(a)

$$\begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

Sol

$$\det(A-\lambda I) = (1-\lambda)((4-\lambda)(1-\lambda)+2)$$
$$-\lambda^3 - 6\lambda^2 - 11\lambda + 6 = (\lambda-1)(\lambda-2)(\lambda-3).$$

Now, when $\lambda = 1$, we have

$$\begin{pmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix} \implies \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Now, when $\lambda = 2$, we have

$$\begin{pmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{pmatrix} \implies \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$

Now, when $\lambda = 3$, we have

$$\begin{pmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{pmatrix} \implies \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

(b)

$$T(x,y,z) = (x-2z, 0, -2x+4z)$$

Sol

The matrix of T is

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$$

Now,

$$\det(A-\lambda I) = \lambda((1-\lambda)(4-\lambda)-4)$$
$$\lambda^3 - 5\lambda^2 = \lambda^2(\lambda-5).$$

Now, when $\lambda = 0$, we have

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix} \implies \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

Now, when $\lambda = 5$, we have

$$\begin{pmatrix} -4 & 0 & -2 \\ 0 & -5 & 0 \\ -2 & 0 & -1 \end{pmatrix} \implies \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

Sol

$$\lambda = 3 \implies \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

(d)

$$T(x, y) = (x + 4y, 2x + 3y).$$

Sol

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$$

$$\lambda = -1 \implies \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\lambda = 5 \implies \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(12) Let C^∞ denote the space of continuous functions in one variable on $\mathbb{R}$ which has continuous derivatives of all orders. For example, $\sin\theta, x^1 + 3x + 1, e^{-x^2} \in C^\infty$. Note that C^∞ is a vector space over $\mathbb{R}$. Let $D : C^\infty \to C^\infty$ be a linear transformation defined as

$$D(f(x)) = f'(x).$$

Find the eigenvalues and the corresponding eigenvectors of D.

Sol
As $D(e^{\lambda x} = \lambda e^x$ implies eigenvalues are all $\lambda \in \mathbb{R}$ and the corresponding eigenvector in C^∞ is $e^{\lambda x}$.

(13) Let $D^2 : C^\infty \to C^\infty$ defined as $D^2(f(x)) = f''(x)$. Find the eigenvectors corresponding the eigenvalue $\lambda = 1$.

Sol
The corresponding eigenvectors are $e^{\lambda x}$, $\cosh x$ and $\sinh x$.

(14) Prove that if λ is an eigenvalue of an invertible matrix A and x is a corresponding eigenvector, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} and x is a corresponding eigenvector.

Sol

$$Ax = \lambda x$$
$$\frac{1}{\lambda}x = A^{-1}x$$

(15) Prove that if λ is an eigenvalue of A, x is a corresponding eigenvector, and s is a scalar, then $\lambda - s$ is an eigenvalue of $A - sI$ and x is a corresponding eigenvector.

Sol

$$\begin{aligned}(A - sI)x &= Ax - sIx \\ &= \lambda Ix - sIx \\ &= (\lambda I - sI)x \\ &= (\lambda - s)Ix \\ &= (\lambda - s)x\end{aligned}$$

(16) Prove that if λ is an eigenvalue of A, x is a corresponding eigenvector, and s is a scalar, then $s\lambda$ is an eigenvalue of sA and x is a corresponding eigenvector.

Sol

$$sAx = s\lambda x$$

(17) Find the eigenvalues and bases for the eigenspaces of

$$\begin{pmatrix} -2 & 2 & 3 \\ -2 & 3 & 2 \\ -4 & 2 & 5 \end{pmatrix}.$$

Use it to find the eigenvalues and eigenspaces of A^{-1}, $A - 3I$ and $3A + 2I$ *(Hint: Use the (14), (15) and (16) to do this question.)*

Sol For A

$$\lambda = 3, 1, 2$$
$$\text{eigenspace} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}$$

The eigenvecor and eigenspace of A^{-1} is

$$\lambda = \frac{1}{3}, 1, \frac{1}{2}$$
$$\text{eigenspace} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}$$

The eigenvecor and eigenspace of $A - 3I$ is

$$\lambda = 0, -2, -1$$
$$\text{eigenspace} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}$$

The eigenvecor and eigenspace of $3A+2I$ is

$$\lambda = 11, 5, 8$$

$$\text{eigenspace} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}$$

(18) Find the matrix P that diagonalizes

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

Sol

$$\lambda = 1, 2$$

$$\text{eigenspace} = \begin{pmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

(19) Let

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}.$$

Find A^{15}.

Sol

$$D = P^{-1}AP \implies A = PDP^{-1}.$$

$$A = \begin{pmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1}$$

$$A^{15} = \begin{pmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}^{15} \begin{pmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1}$$

$$A^{15} = \begin{pmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^{15} & 0 \\ 0 & 0 & 2^{15} \end{pmatrix} \begin{pmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -32766 & 0 & -65534 \\ 32767 & 32768 & 32767 \\ 32767 & 0 & 65535 \end{pmatrix}$$

(20) Show that the following matrix is not diagonalizable.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{pmatrix}.$$

Sol

Eigenvalues are $\lambda = 1, 2$. When $\lambda = 1$, we have

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & 5 & 1 \end{pmatrix} \qquad \text{giving 1 eigenvector.}$$

When $\lambda = 2$, we have

$$\begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ -3 & 5 & 0 \end{pmatrix} \qquad \text{giving 1 eigenvector.}$$

As there are not 3 linearly independent vectors so A is no diagonalizable.

(21) Find the eigenvalues of A^7 where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{pmatrix}.$$

Sol

$$\lambda = 1, 2^7, 2^7.$$

(22) Show that the following matrices are similar.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Sol

Eigenvalues and eigenspace for A is

$$\lambda = 3, 0$$

$$\text{eigenspace} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

Now,

$$B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = P^{-1}AP = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

(23) Show that the following matrices are not similar.

$$A = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix}$$

Sol Since $\det(A) = -1 \neq \det(B) = -2$, therefore they are not similar.

True-False Exercises

TF. In parts (a)–(f) determine whether the statement is true or false, and justify your answer.

(a) If A is a square matrix and $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero scalar λ, then $\mathbf{x}$ is an eigenvector of A.

(b) If λ is an eigenvalue of a matrix A, then the linear system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(c) If the characteristic polynomial of a matrix A is $p(\lambda) = \lambda^2 + 1$, then A is invertible.

(d) If λ is an eigenvalue of a matrix A, then the eigenspace of A corresponding to λ is the set of eigenvectors of A corresponding to λ.

(e) The eigenvalues of a matrix A are the same as the eigenvalues of the reduced row echelon form of A.

(f) If 0 is an eigenvalue of a matrix A, then the set of columns of A is linearly independent.

True-False Exercises

TF. In parts (a)–(i) determine whether the statement is true or false, and justify your answer.

(a) An $n \times n$ matrix with fewer than n distinct eigenvalues is not diagonalizable.

(b) An $n \times n$ matrix with fewer than n linearly independent eigenvectors is not diagonalizable.

(c) If A and B are similar $n \times n$ matrices, then there exists an invertible $n \times n$ matrix P such that $PA = BP$.

(d) If A is diagonalizable, then there is a unique matrix P such that $P^{-1}AP$ is diagonal.

(e) If A is diagonalizable and invertible, then A^{-1} is diagonalizable.

(f) If A is diagonalizable, then A^T is diagonalizable.

(g) If there is a basis for R^n consisting of eigenvectors of an $n \times n$ matrix A, then A is diagonalizable.

(h) If every eigenvalue of a matrix A has algebraic multiplicity 1, then A is diagonalizable.

(i) If 0 is an eigenvalue of a matrix A, then A^2 is singular.

THEOREM 5.1.5 Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent.

(*a*) *A is invertible.*

(*b*) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(*c*) *The reduced row echelon form of A is* I_n.

(*d*) *A is expressible as a product of elementary matrices.*

(*e*) $A\mathbf{x} = \mathbf{b}$ *is consistent for every* $n \times 1$ *matrix* $\mathbf{b}$.

(*f*) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every* $n \times 1$ *matrix* $\mathbf{b}$.

(*g*) $\det(A) \neq 0$.

(*h*) *The column vectors of A are linearly independent.*

(*i*) *The row vectors of A are linearly independent.*

(*j*) *The column vectors of A span* R^n.

(*k*) *The row vectors of A span* R^n.

(*l*) *The column vectors of A form a basis for* R^n.

(*m*) *The row vectors of A form a basis for* R^n.

(*n*) *A has rank n.*

(*o*) *A has nullity* 0.

(*p*) *The orthogonal complement of the null space of A is* R^n.

(*q*) *The orthogonal complement of the row space of A is* $\{\mathbf{0}\}$.

(*r*) *The kernel of* T_A *is* $\{\mathbf{0}\}$.

(*s*) *The range of* T_A *is* R^n.

(*t*) T_A *is one-to-one.*

(*u*) $\lambda = 0$ *is not an eigenvalue of A.*

THEOREM 5.2.1 *If A is an $n \times n$ matrix, the following statements are equivalent.*

(*a*) *A is diagonalizable.*

(*b*) *A has n linearly independent eigenvectors.*

THEOREM 5.2.2

(*a*) *If* $\lambda_1, \lambda_2, \ldots, \lambda_k$ *are distinct eigenvalues of a matrix A, and if* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ *are corresponding eigenvectors, then* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *is a linearly independent set.*

(*b*) *An* $n \times n$ *matrix with n distinct eigenvalues is diagonalizable.*

THEOREM 5.2.3 *If k is a positive integer,* λ *is an eigenvalue of a matrix A, and* $\mathbf{x}$ *is a corresponding eigenvector, then* λ^k *is an eigenvalue of* A^k *and* $\mathbf{x}$ *is a corresponding eigenvector.*

There is some terminology that is related to these ideas. If λ_0 is an eigenvalue of an $n \times n$ matrix A, then the dimension of the eigenspace corresponding to λ_0 is called the ***geometric multiplicity*** of λ_0, and the number of times that $\lambda - \lambda_0$ appears as a factor in the characteristic polynomial of A is called the ***algebraic multiplicity*** of λ_0. The following theorem, which we state without proof, summarizes the preceding discussion.

THEOREM 5.2.4 Geometric and Algebraic Multiplicity

If A is a square matrix, then:

(*a*) *For every eigenvalue of A, the geometric multiplicity is less than or equal to the algebraic multiplicity.*

(*b*) *A is diagonalizable if and only if the geometric multiplicity of every eigenvalue is equal to the algebraic multiplicity.*