Math 6051/3051: Recitation 11 ☺

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Do all of the following problems.

(1) Find all $x \in \mathbb{R}$ that satisfy the equation $|x+1| + |x-2| = 7$.
Sol:

$$\begin{aligned} x+1+x-2 = 7 &\implies x = 4 \\ -(x+1)-(x-2) = 7 &\implies x = -3 \\ (x+1)-(x-2) = 7 &\implies \text{No solutions} \\ -(x+1)+(x-2) = 7 &\implies \text{No solutions} \end{aligned}$$

(2) Show that if A and B are bounded set of $\mathbb{R}$, then $A \cup B$ is also bounded. Furthermore, show that $\sup\{A \cup B\} = \sup\{\sup A, \sup B\}$.
Sol:
Let A and B are bounded set. If M and N are upper bounds of A and B respectively, then $\max\{A, B\}$ is an upped bound of $A \cup B$. In particular, $\sup\{A \cup B\} \leq \sup\{\sup A, \sup B\}$. For the converse, let $\sup\{\sup A, \sup B\} = \sup A$. Now if M is an upper bound of $A \cup B$ it implies $\sup A \leq M$. Thus, $\sup\{\sup A, \sup B\} \leq \sup\{A \cup B\}$.

(3) Show that $\lim\left(\frac{\sqrt{n}}{n+1}\right) = 0$.
Sol:

$$0 \leq \lim_{n\to\infty} \left|\frac{\sqrt{n}}{n+1}\right| \leq \lim_{n\to\infty} \left|\frac{\sqrt{n}}{n}\right| = \lim_{n\to\infty} \left|\frac{1}{\sqrt{n}}\right| = 0$$

So, $\lim\left(\frac{\sqrt{n}}{n+1}\right) = 0$.

(4) Let $x_1 = a > 0$ and $x_{n+1} = x_n + \frac{1}{x_n}$ for $n \in \mathbb{N}$. Determine whether the sequence (x_n) converges or diverges.
Sol:
If it converges, then it limit would be the solution of the equation $x = x + \frac{1}{x}$. But since, this equation possesses no solutions, so, x_n does not converge.

(5) Show directly that a bounded, monotone increasing sequence is a Cauchy sequence. *(Do not use the Monotone Converging Theorem)*
Sol:
Let (x_n) be a bounded, monotone increasing sequence and let M be its supremum. Let $\epsilon > 0$. Then there exists N such that $M - \epsilon < x_N \le M$. Now since (x_n) is monotone increasing, it implies $M - \epsilon < x_n \le M$ for all $n \ge N$. So, for $m, n > N$, we have $x_m, x_n \in (M - \epsilon, M]$. In particular, $|x_m - x_n| < \epsilon$. Thus, (x_n) is Cauchy.

(6) Give an example of a sequence that is not Cauchy, but $|x_{n+2} - x_n| = 0$ for all $n \ge 1$.
Sol:
$\{0, 2, 0, 2, 0, 2, \cdots\}$

(7) Show that series $\sum_{n=1}^{\infty} \cos\{n\}$ is divergent.
Sol:
Since $\lim \cos n = DNE$, it implies $\sum \cos n$ diverges.

(8) If $\sum a_n$ converges where $a_n > 0$ for all $n \in \mathbb{N}$, and if $b_n = \frac{a_1 + \cdots + a_n}{n}$ for $n \in \mathbb{N}$, then show that $\sum b_n$ is always divergent.
Sol:
Observe that

$$\begin{aligned}\sum b_n &= a_1 + \frac{a_1 + a_2}{2} + \frac{a_1 + a_2 + a_3}{3} + + \frac{a_1 + a_2 + a_3 + a_4}{4} + \cdots \\ &= a_1\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + a_2\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + a_3\left(\frac{1}{3} + \frac{1}{4}\right) + \cdots \\ &\ge a_1 \sum_{n=1}^{\infty} \frac{1}{n}.\end{aligned}$$

Since $\sum \frac{1}{n}$ diverges, it implies $\sum b_n$ diverges.

(9) Let $K > 0$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy the condition $|f(x) - f(y)| \le K|x - y|$ for all $x, y \in \mathbb{R}$. Show that f is uniformly continuous on $\mathbb{R}$. Argue that the converse might not be true.
Sol:
Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{K}$. Then for any $x, y \in \mathbb{R}$, we have

$$|f(x) - f(y)| \le K|x - y| < K \cdot \frac{\epsilon}{K} = \epsilon.$$

Thus, f is uniformly continuous. The function $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$ but it does not satisfy the above condition. (CHECK IT) Because if it was the case that for some K, we had $|f(x) - f(y)| \le K|x - y|$, then

$$\begin{aligned}|\sqrt{x} - \sqrt{y}| &\le K|x - y| \\ \frac{|x - y|}{\sqrt{x} + \sqrt{y}} &\le K|x - y| \\ \frac{1}{\sqrt{x} - \sqrt{y}} &\le K.\end{aligned}$$

But this inequality does not hold on $[0, \infty)$.

(10) A function $f : \mathbb{R} \to \mathbb{R}$ is said to be additive if $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Prove that if f is continuous at a point x_0 then it is continuous on $\mathbb{R}$.
Sol:
Let f be continuous at x_0. Without loss of generality, assume $x_0 = 0$. Furthermore, observe that $f(x) = f(x + 0) = f(x) + f(0)$, we must have $f(0) = 0$. Furthermore, $f(0) = f(x - x) = f(x) + f(-x)$

implies $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Let $a_0 \in \mathbb{R}$. Now since f is continuous at 0, for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|x| < \delta \implies |f(x)| < \epsilon.$$

So, for any $a \in \mathbb{R}$ such that

$$|a - a_0| < \delta \implies |f(a - a_0)| < \epsilon.$$

But then since $f(a - a_0) = f(a) - f(a_0)$, we have f is continuous at a_0. As a_0 was an arbitrary element, it implies f is continuous on $\mathbb{R}$.

(11) Argue that if $f : [a, b] \to \mathbb{R}$ is a continuous function, then you can extend if to a function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ such that $\tilde{f}|_{[a,b]} = f$. (*Some pictures might help*)
Sol:
Since f is continuous on $[a, b]$, it implies f is uniformly continuous. Define the function $\tilde{f}$ as

$$\tilde{f}(x) = \begin{cases} f(a), & x \le a \\ f(x), & a \le x \le b \\ f(b), & b \le x. \end{cases}.$$

Clearly $\tilde{f}$ is continuous on $[a, b]$ and $\tilde{f}|_{[a,b]} = f$.

(12) Show that if f is continuous on $[0, \infty)$ and uniformly continuous on $[a, \infty)$ for some $a > 0$. Then f is uniformly continuous on $[0, \infty)$.
Sol:
Given that f is uniformly continuous on $[a, \infty)$. Furthermore, since f is continuous on $[0, a]$, it implies f is also uniformly continuous on $[0, a]$. Combining these facts, it implies f is uniformly continuous on $[0, \infty)$.

(13) If $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $\mathbb{R}$, and $|f(x)| \ge k > 0$ for all $x \in \mathbb{R}$, show that $\frac{1}{f}$ is also uniformly continuous on $\mathbb{R}$.
Sol:
Note that for the function $g(x) = \frac{1}{f(x)}$, we would have $|g(x)| \le k$. Let $\epsilon > 0$. Note that

$$\begin{aligned} |g(x) - g(y)| &= \left| \frac{f(y) - f(x)}{f(x)f(y)} \right| \\ &\le \frac{1}{k^2}|f(y) - f(x)|. \end{aligned}$$

Since, f is continuous, there exits $\delta > 0$ such that $|f(x) - f(y)| < k^2\epsilon$ whenever $|x - y| < \delta$. For this δ, we would have

$$|g(x) - g(y)| < \epsilon.$$

Thus, $g = 1/f$ is uniformly continuous.

(14) Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = 2x$ for x rational, and $g(x) = x + 3$ for x irrational. Find all points at which g is continuous.
Sol:
Continuous only on $x = 3$.

(15) Give an example of a function on $\mathbb{R}$ that is continuous at exactly 3 points.
Sol:
$f(x) = |2x|$ when x is rational, and $f(x) = x^2$ when x is irrational. The function is continuous on $x = -2, 0, 2$.

(16) Prove or give a counterexample that the product of two uniformly continuous functions on $\mathbb{R}$ are not uniformly continuous.

Sol:

Consider the function $f(x) = x$ on $\mathbb{R}$. The function $g(x) = f(x)f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.(CHECK IT). Suppose on contrary that x^2 is uniformly continuous on $\mathbb{R}$. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that whenever $|x - y| < \delta$, we have $|x^2 - y^2| < \epsilon$. Now, since $|x^2 - y^2| = |x - y||x + y|$, by uniform continuity we have that $|x + y|$ is bounded on $\mathbb{R}$, a contradiction.