# Math 6091/3091: Recitation 10

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Do **all** of the following problems.

(1) **(2 points)**

(a) Show that if $A$ and $B$ are symmetric matrices, then $A + B$ is symmetric.

**Sol:**

$$(A + B)^T = A^T + B^T = A + B.$$

(b) Show that for any $A \in M_{n\times n}(\mathbb{R})$, $A + A^T$ and $AA^T$ are symmetric.

**Sol:**

$$(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$$

Note that $(AB)^T = B^T A^T$. So,

$$(AA^T)^T = (A^T)^T A^T = AA^T.$$

(2) **(2 points)** A matrix $A \in M_{n\times n}(\mathbb{R})$ is said to be normal if $AA^T = A^T A$.

(a) Show that if $A$ is symmetric then $A$ is normal.

**Sol:**
Obvious

$$AA^T = A^T A$$

(b) Find an example of a $2 \times 2$ matrix that is normal but not symmetric.

**Sol:**

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(c) Show that if $A$ is normal then $||Ax|| = ||A^T x||$. Also, show that $A - cI$ is normal for all $c \in \mathbb{R}$.

**Sol:**
As

$$||Ax||^2 = \langle Ax, Ax\rangle = \langle x, A^T Ax\rangle = \langle x, AA^T x\rangle = \langle A^T x, A^T x\rangle = ||A^T x||^2.$$

$$\begin{aligned}(A - cI) * (A - cI)^T = (A - cI) * (A^T - cI) &= AA^T - cIA - cIA^T + c^2 I \\ &= A^T A - cIA - cIA^T + c^2 I \\ &= (A - cI)^T * (A - cI).\end{aligned}$$

(3) **(3 points)**

(a) Suppose $\alpha$ is an orthonormal basis of $\mathbb{R}^n$, and let $Q$ be the change of basis matrix from the standard basis to $\alpha$. Show that $Q$ satisfies $Q^T = Q^{-1}$.

**Sol:**
Since $Q$ is the change of basis matrix from the standard basis, the columns of $Q$ consists of the orthonormal basis vectors of $\alpha$. So, the columns of $Q$ are orthogonal to each other. Now, since $Q * Q^T$ amounts of multiplying all the vectors of the basis $\alpha$ with each other, so $Q * Q^T = I$. Similarly, $Q^T * Q = I$. Therefore, $Q^T = Q^{-1}$.

(b) Show conversely that if $Q \in M_{n\times n}(\mathbb{R})$ and $Q^T = Q^{-1}$, then the columns of $Q$ forms an orthonormal basis.

**Sol:**
If $Q^T = Q^{-1}$, it implies $QQ^T = Q^TQ = I$. Let $v_1, \cdots, v_n$ be column of $Q$. The existence of $Q^{-1}$ suggests $v_i$ are linearly independent. The fact $QQ^T = Q^TQ = I$ implies

$$\langle v_i, v_j \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

So, the vectors are orthonormal to each other. Observe that, it is orthonormal because the inner product is 1 rather than any other number.

(4) **(3 points)** Verify the spectral theorem for each of the following symmetric matrices, by finding an orthonormal basis of the appropriate vector space, the change of basis matrix to this basis, and the spectral decomposition.

(a) $\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$

**Sol:**
Eigenvalues are 5 and $-1$ and eigenvectors are $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ respectively. They both are orthogonal. So, we have

$$\begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

(b) $\begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix}$

**Sol:**
Eigenvalues are $\frac{5-\sqrt{13}}{2}, \frac{5+\sqrt{13}}{2}$ and eigenvectors are $E_1 = (1, -\frac{-3-\sqrt{13}}{2})$ and $E_2(1, -\frac{-3+\sqrt{13}}{2})$ respectively. They both are orthogonal. So, we have

$$\begin{pmatrix} \frac{5-\sqrt{13}}{2} & 0 \\ 0 & \frac{5+\sqrt{13}}{2} \end{pmatrix} = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} E_1 & E_2 \end{pmatrix}$$