# Math 6091/3091: Recitation 11

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(1) **(2 points)** Determine whether the given subspace $W \subseteq V$ is invariant under the given linear map $T : V \to V$.

$$V = \mathbb{R}^3, \quad T : V \to V \text{ defined by } \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad W = \text{Span}\{(1,-1,1),(1,2,1)\}.$$

**Sol:** Let $v_1 = (1,-1,1)$.

$$T(v_1) = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & -1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$$

To check if $T(v_1) \in W$, we attempt to express it as a linear combination of the basis vectors:

$$\begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} = a \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

This gives the system:

$$\begin{cases} a + b = -1 \\ -a + 2b = -3 \\ a + b = 1 \end{cases}$$

The first and third equations contradict each other, so no such $a, b$ exist. Therefore, $W$ is not stable.

(2) (**1 point**) Compute the inverse of the matrix $\begin{pmatrix} 1 & 4 \\ 3 & 8 \end{pmatrix}$ using characteristic polynomial $p(x) = (-1)^n\lambda^n + \cdots + a_2\lambda + a_1$ and the formula

$$A^{-1} = \frac{-1}{\det(A)}((-1)^n A^{n-1} + \cdots + a_2 A + a_1 I).$$

**Sol:** Characteristic polynomial is

$$p(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 4 \\ 3 & 8-\lambda \end{pmatrix} = (1-\lambda)(8-\lambda) - 12$$
$$= \lambda^2 - 9\lambda - 4$$

So,

$$A^2 - 9A - 4I = 0 \implies A^{-1} = \frac{1}{4}(A - 9I).$$

$$A^{-1} = \begin{pmatrix} -2 & 1 \\ 3/4 & -1/4 \end{pmatrix}$$

(3) **(2 points)** Consider the nilpotent matrix

$$A = \begin{pmatrix} 0 & 2 & 3 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $x = (1,1,1,3)$. Find $C(x) = \text{Span}\{A^{k-1}(x), A^{k-2}(x), \cdots, Ax, x\}$ where $A^r = 0$ for all $r \geq k$.

**Sol:** Observe that

$$A^2 = \begin{pmatrix} 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad A^3 = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad A^4 = 0$$

Now,

$$A^0 x = x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix}$$

$$Ax = \begin{pmatrix} 0 & 2 & 3 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 3 \\ 0 \end{pmatrix}$$

$$A^2 x = \begin{pmatrix} 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 17 \\ 3 \\ 0 \\ 0 \end{pmatrix}$$

$$A^3 x = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

So,

$$C(x) = \text{Span}\left\{ \begin{pmatrix} 6 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 17 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ 4 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix} \right\}$$

(4) **(2 points)** Let $N : \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$A = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Find a basis $\beta$ such that $[N]_\beta^\beta$ is in the canonical form.

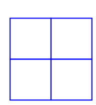
**Sol:** Eigenvectors of $A$ are $E_1$ and $E_3$. Furthemore, observe that

$$\alpha_1 = C\left(\frac{1}{2}E_2\right) = \left\{E_1, \frac{1}{2}E_2\right\}$$

and

$$\alpha_2 = C\left(E_4\right) = \left\{E_3, E_4\right\}.$$

So, $\alpha_1 \cup \alpha_2$ forms a basis of $\mathbb{R}^4$. The Tableu corresponding to this is



So, with this basis, the matrix representation becomes:

$$[N]_\beta^\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(5) **(3 points)** Let $T : \mathbb{C}^9 \to \mathbb{C}^9$ be a linear mapping and assume that the characteristic polynomial of $T$ is $((2-i)-\lambda)^4(3-\lambda)^5$. Assume that

$$\dim(\mathrm{Ker}(T-(2-i)I)) = 3$$
$$\dim(\mathrm{Ker}((T-(2-i)I)^2)) = 4$$

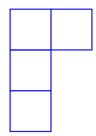
and

$$\dim(\mathrm{Ker}(T-3I)) = 2$$
$$\dim(\mathrm{Ker}(T-3I)^2) = 4$$
$$\dim(\mathrm{Ker}(T-3I)^3) = 5$$

Find the Jordan Canonical Form of $T$.

**Sol:**

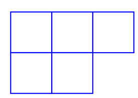
Tableau corresponding to the eigenvalue $2-i$ is



This gives Jordan block

$$J_1 = \begin{pmatrix} 2-i & 1 & 0 & 0 \\ 0 & 2-i & 0 & 0 \\ 0 & 0 & 2-i & 0 \\ 0 & 0 & 0 & 2-i \end{pmatrix}$$

Tableau corresponding to the eigenvalue 3 is



This gives Jordan block

$$J_2 = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

Therefore, the Jordan Canonical form is

$$\begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} = \begin{pmatrix} 2-i & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2-i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2-i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2-i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$