Math 6091/3091: Recitation 3

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Do any **all** of the following problems.

(1) **(2 points)** Let V and W be vector spaces over a field $\mathbf{F}$ and let $T : V \longrightarrow W$ be a linear transformation. Show that the image of T is a subspace of W i.e. the set

$$T(V) := \{w \in W \mid \text{there exists } v \in V \text{ such that } T(v) = w\}$$

is a subspace of W.

Sol:
Let $w_1, w_2 \in T(V)$. Then there exists $v_1, v_2 \in V$ such that $T(v_1) = w_1$ and $T(v_2) = w_2$. Now for any $\alpha, \beta \in F$, the element $\alpha w_1 + \beta w_2 \in T(V)$ since

$$T(\alpha v_1 + \beta v_2) = \alpha w_1 + \beta w_2.$$

Thus, $T(V)$ is a subspace in W.

(2) **(3 points)** Let $V = \mathbb{R}^2$ and $W = P_3(\mathbb{R})$. If $T : V \longrightarrow W$ is a linear transformation defined by

$$T((1,1)) = x + x^2 \quad \text{and} \quad T((3,0)) = x - x^3.$$

(a) Find $T(7,1)$.

Sol:

$$\begin{aligned} T(7,1) &= T(2(3,0) + (1,1)) \\ &= 2T(3,0) + T(1,1) \\ &= 2(x + x^2) + (x - x^3) \\ &= 3x + 2x^2 - x^3 \end{aligned}$$

(b) Find $v \in V$ such that $T(v) = -x^3 - x^2$.

Sol:
As $x - x^3 - (x + x^2) = -x^3 - x^2$, we have

$$T((3,0) - (1,1)) = T(2,-1) = -x^3 - x^2.$$

(c) Is T one-to-one? Is T onto?

Sol:
We compute the Kernel of T. Let $\alpha, \beta \in F$. Consider the following:

$$\begin{aligned} &T(\alpha(3,0) + \beta(1,1)) = 0 \\ \Longrightarrow\ &\alpha T(3,0) + \beta T(1,1) = 0 \\ \Longrightarrow\ &\alpha(x + x^2) + \beta(x - x^3) = 0 \\ \Longleftrightarrow\ &\alpha = \beta = 0 \end{aligned}$$

So, $Ker(T) = 0$, thus, T is one to one. Now as $\dim W > \dim V$, T cannot be onto.

(3) **(2 points)** Let $V = \mathbb{R}^3$ and $W = \mathbb{R}^4$, and let $T : V \longrightarrow W$ be a linear transformation defined as

$$T((v_1, v_2, v_3)) = (v_1 - v_2, v_2 - v_3, v_1 + v_2 - v_3, v_3 - v_1).$$

What is the matrix of T with respect to the standard basis $\{(1,0,0), (0,1,0), (0,0,1)\}$.

Sol:

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

(4) **(3 points)** Let $V = \mathbb{R}^4$ and $W = \mathbb{R}^3$, and let $T : V \longrightarrow W$ be a linear transformation defined as

$$T((v_1, v_2, v_3, v_4)) = (v_1 - v_2, v_2 - v_3 + v_4, v_1 + v_2 - v_3 + v_4).$$

(a) **(2 point)** Find the bases for the kernel of T.

Sol:

$$T((v_1, v_2, v_3, v_4)) = 0$$
$$(v_1 - v_2, v_2 - v_3 + v_4, v_1 + v_2 - v_3 + v_4) = (0,0,0)$$

So, we have

$$v_1 - v_2 = 0 \implies v_1 = v_2$$
$$v_2 - v_3 + v_4 = 0 \implies v_1 = v_3 - v_4$$
$$v_1 + v_2 - v_3 + v_4 = 0 \implies 2v_1 = v_3 - v_4$$

From this, we have $v_1 = 2v_1$ which implies $v_1 = 0$. This also indicates $v_2 = 0$ and $v_3 - v_4 = 0 \implies v_3 = v_4$. So, the Kernel is

$$Ker(T) = Span\{(0,0,1,1)\}.$$

(b) **(1 point)** Find the basis of the image of T.

Sol:
Since $\dim V - \dim Ker(T) = 4 - 1 = 3 = \dim W$, we have T is onto. So, any basis of $W = \mathbb{R}^3$ would work. In particular, $\{(1,0,0), (0,1,0), (0,0,1)\}$.