Math 6091/3091: Recitation 5

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Do any **all** of the following problems.

(1) **(2 points)** For each of the following matrices, defining linear maps T between vector spaces of the appropriate dimensions, find bases for $Ker(T)$.

(a) **(1 point)**

$$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

Sol:

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

So, $Ker = 0$ and the basis for kernel $B = \emptyset$.

(b) **(1 point)**

$$\begin{bmatrix} -1 & 2 & 2 \\ 2 & -4 & 2 \end{bmatrix}$$

Sol:

$$\begin{bmatrix} -1 & 2 & 2 & 0 \\ 2 & -4 & 2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

So, $Ker = span\{(2, 1, 0)\}$.

(2) **(2 points)** Consider the following maps:

$$T_1 : (v_1, v_2, v_3) = T(v_1 + v_2, v_2 + v_3, v_1 + 3v_3, v_1 - v_2 - 2v_3)$$

$$T_2 : (v_1, v_2, v_3, v_4) = T(v_1 + v_2, v_3 - v_4).$$

(a) Find $T_2 \circ T_1$.

Sol:
We know

$$A_{T_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -1 & -2 \end{bmatrix} \text{ and } A_{T_2} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Hence,

$$A_{T_2 \circ T_1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 5 \end{bmatrix}$$

(b) Compute $T_2 \circ T_1(1, 1, 1)$.

Sol:

$$T_2 \circ T_1(1,1,1) = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

(3) **(5 points)** Let $V = \mathbb{R}^3$ and $W = \mathbb{R}^3$, and let $T : V \longrightarrow W$ be a linear transformation. Let

$$\alpha = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ be basis of } V \text{ and } \beta = \left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\} \text{ be basis of } W$$

Suppose T is given by

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = -\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

(a) **(2 point)** Write $v = \begin{bmatrix} 5 \\ 5 \\ 20 \end{bmatrix}$ in terms of the basis α.

Sol:

$$\begin{bmatrix} 1 & 0 & 1 & 5 \\ 1 & 1 & 0 & 5 \\ 0 & 1 & 1 & 20 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 10 \end{bmatrix}.$$

So, $[v]_\alpha = \begin{bmatrix} -5 \\ 10 \\ 10 \end{bmatrix}$ i.e. $-5v_1 + 10v_2 + 10v_3 = v$.

(b) **(3 point)** Use the above to compute $T\left(\begin{bmatrix} 5 \\ 5 \\ 20 \end{bmatrix}\right)$

Sol:
We know that,

$$\left[T\left(\begin{bmatrix} 5 \\ 5 \\ 20 \end{bmatrix}\right)\right]_\beta = T_\alpha^\beta [v]_\alpha.$$

As $T_\alpha^\beta = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & -2 \\ 2 & 0 & 1 \end{bmatrix}$, we have

$$T_\alpha^\beta [v]_\alpha = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & -2 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ 10 \\ 10 \end{bmatrix} = \begin{bmatrix} -5 \\ -10 \\ -20 \end{bmatrix}$$

So,

$$\left[T\left(\begin{bmatrix}5\\5\\20\end{bmatrix}\right)\right]_\beta = \begin{bmatrix}-5\\-10\\-20\end{bmatrix}_\beta.$$

Hence,

$$T\left(\begin{bmatrix}5\\5\\20\end{bmatrix}\right) = -5w_1 - 10w_2 - 20w_3 = \begin{bmatrix}-40\\-45\\-20\end{bmatrix}$$