Math 6091/3091: Recitation 6

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Do **all** of the following problems. **Here** $P_{B\to B'}$ **denotes** $[I]_B^{B'}$ **from the book.**

(1) Consider the basis $B = \{u_1, u_2\}$ and $B' = \{u'_1, u'_2\}$ of $\mathbb{R}^2$ where

$$u_1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 4 \\ -1 \end{pmatrix}, \quad u'_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad u'_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

(a) Find $P_{B\to B'}$ and $P_{B'\to B}$.

Sol:

$$P_{B\to B'} = rref\begin{pmatrix} 1 & -1 & \vdots & 2 & 4 \\ 3 & -1 & \vdots & 2 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -2.5 \\ -2 & -6.5 \end{pmatrix}$$

$$P_{B'\to B} = rref\begin{pmatrix} 2 & 4 & \vdots & 1 & -1 \\ 2 & -1 & \vdots & 3 & -1 \end{pmatrix} = \begin{pmatrix} 1.3 & -0.5 \\ -0.4 & 0 \end{pmatrix}$$

(b) Find $[w]_B$ where $w = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$. Use $P_{B\to B'}$ to compute $[w]_{B'}$.

Sol:
To find $[w]_B$, we do the following

$$\begin{pmatrix} 2 & 4 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \end{pmatrix} \implies [w]_B = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1.7 \\ 1.6 \end{pmatrix}$$

So,

$$[w]_{B'} = P_{B\to B'}[w]_B = \begin{pmatrix} 0 & -2.5 \\ -2 & -6.5 \end{pmatrix}\begin{pmatrix} -1.7 \\ 1.6 \end{pmatrix} = \begin{pmatrix} -4 \\ -7 \end{pmatrix}$$

(2) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be the linear transformation defined by

$$T(u_1) = v_1 - 2v_3, \quad T(u_2) = 3v_1 + v_2 - v_3$$

where

$$u_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}; \qquad v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Find the matrix for the linear transformation T with respect to the standard bases i.e. find the formula for T.

Sol:

$$[T]_\alpha^\beta = \begin{pmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{pmatrix}$$

Let e_2 and e_3 denote the standard basis of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. So,

$$[I]^{\alpha}_{e_2} = rref\begin{pmatrix} 3 & 5 & \vdots & 1 & 0 \\ 1 & 2 & \vdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$$

$$[I]^{e_3}_{\beta} = rref\begin{pmatrix} 1 & 0 & 0 & \vdots & 1 & -1 & 0 \\ 0 & 1 & 0 & \vdots & 0 & 2 & 1 \\ 0 & 0 & 1 & \vdots & -1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{pmatrix}$$

So,

$$\begin{aligned} [T]^{e_3}_{e_2} &= [I]^{e_3}_{\beta}[T]^{\beta}_{\alpha}[I]^{\alpha}_{e_2} \\ &= \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{pmatrix} \end{aligned}$$

So,

$$T(x, y) = \begin{pmatrix} y \\ -5x + 13y \\ -7x + 16y \end{pmatrix}$$

(3) Find the determinant of the following matrices:

$$\begin{pmatrix} 1 & -3 \\ 6 & 7 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{pmatrix}$$

Sol:

$$\begin{vmatrix} 1 & -3 \\ 6 & 7 \end{vmatrix} = 25 \qquad \text{and} \qquad \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = 152$$

(4) Show that the matrices

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}$$

commute (i.e. $AB = BA$) if and only if

$$\begin{vmatrix} b & a - c \\ e & d - f \end{vmatrix} = 0$$

Sol:
Note that

$$\begin{aligned} AB &= BA \\ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} d & e \\ 0 & f \end{pmatrix} &= \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \\ \begin{pmatrix} ad & ae + bf \\ 0 & cf \end{pmatrix} &= \begin{pmatrix} ad & bd + ec \\ 0 & cf \end{pmatrix} \\ ae + bf &= bd + ec \end{aligned}$$

Now

$$\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = 0$$

$$b(d-f) - e(a-c) = 0$$

$$bd - bf - ea + ec = 0$$

$$ea + bf = bd + ec$$

(5) Prove that the equation of the line through the distinct points (a_1, b_1) and (a_2, b_2) can be written as

$$\begin{vmatrix} x & y & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix} = 0$$

Sol:

$$\begin{vmatrix} x & y & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix} = x(b_1 - b_2) - y(a_1 - a_2) + 1(a_1b_2 - b_1a_2)$$

So,

$$x(b_1 - b_2) - y(a_1 - a_2) + (a_1b_2 - b_1a_2) = 0$$

$$y(a_1 - a_2) = x(b_1 - b_2) + (a_1b_2 - b_1a_2)$$

$$y = \left(\frac{b_1 - b_2}{a_1 - a_2}\right) x + \frac{a_1b_2 - b_1a_2}{a_1 - a_2}$$

Which is the equation of the line.