Math 6091/3091: Recitation 8

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Do **all** of the following problems.

(1) **(2 points)** Let $x = (1, 2, 3, 1)$ and $y = (1, 2, -4, 1)$.

(a) Compute $\langle x, y\rangle$, $||x||$ and $||y||$.

Sol:

$$\langle x, y\rangle = -6, \quad ||x|| = \sqrt{15}, \quad ||y|| = \sqrt{22}$$

(b) Find the angle between x and y.

Sol:

$$\theta = \cos^{-1}\left(\frac{-6}{\sqrt{15}\sqrt{22}}\right)$$

(2) **(2 points)** Show that for any vector $x, y \in \mathbb{R}^n$ and any scalar $c \in \mathbb{R}$, we have

$$\langle x - cy, x - cy\rangle = ||x||^2 - 2c\langle x, y\rangle + c^2||y||^2.$$

Sol:

$$\begin{aligned}
\langle x - cy, x - cy\rangle &= \langle x, x - cy\rangle + \langle -cy, x - cy\rangle \\
&= \langle x, x\rangle + \langle x, -cy\rangle + \langle -cy, x\rangle + \langle -cy, -cy\rangle \\
&= \langle x, x\rangle - 2c\langle x, y\rangle + c^2\langle y, y\rangle \\
&= ||x||^2 - 2c\langle x, y\rangle + c^2||y||^2.
\end{aligned}$$

(3) **(3 points)** Let V be a vector space. An *inner product* on V is any mapping $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$ with the following three properties:

(i) For all vectors $x, y, z \in V$ and $c \in \mathbb{R}$, $\langle cx + y, z\rangle = c\langle x, z\rangle + \langle y, z\rangle$.
(ii) For all $x, y \in V$, we have $\langle x, y\rangle = \langle y, x\rangle$.
(iii) For all $x \in V$, $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0$ if and only if $x = 0$.

Show that the mapping $\langle\ ,\ \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\langle (x_1, x_2), (y_1, y_2)\rangle = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

is an inner product.

Sol:

We first show (i). Note that the above inner product can be written as

$$\langle x, y\rangle = x^T Ay$$

where $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Now for $x, y, z \in V$ and $a \in \mathbb{R}$, we have

$$\langle ax + y, z\rangle = (ax + y)^T Az = (ax^T + y^T)Az = ax^T Az + ay^T Az = a\langle x, z\rangle + \langle y, z\rangle.$$

Now to prove (ii), note that

$$\langle x, y\rangle = x^T Ay = x_1(2y_1 + y_2) + x_2(y_1 + y_2) = 2x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2$$

and

$$\langle y, x\rangle = y^T Ax = y_1(2x_1 + x_2) + y_2(x_1 + x_2) = 2x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2.$$

So, $\langle x, y\rangle = \langle y, x\rangle$.

Finally for (iii), $\langle x, x\rangle = 2x_1^2 + 2x_1x_2 + x_2^2$, which is at least zero as it is the sum of squares. Furthermore it is equal to 0 if and only if both x_1 and x_2 equals to 0.

(4) **(3 points)** Find an orthonormal basis of the subspace W in $\mathbb{R}^5$ defined by

$$x_1 + 2x_2 - x_3 + x_5 = 0,$$
$$3x_1 + x_3 + 3x_4 = 0.$$

Sol:

$$\begin{pmatrix} 1 & 2 & -1 & 0 & 1 \\ 3 & 0 & 1 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} 3 & 0 & 1 & 3 & 0 \\ 1 & 2 & -1 & 0 & 1 \end{pmatrix}$$
$$\to \begin{pmatrix} 3 & 0 & 1 & 3 & 0 \\ 0 & 6 & -4 & -3 & 3 \end{pmatrix}$$

So, the kernel is

$$\left\{ u_1 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, u_2 = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, u_3 = \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

which is the basis of W. To produce an orthogonal basis, let

$$v_1 = u_1.$$

Now,

$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{||v_1||^2} v_1 = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{-2}{5} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1.6 \\ 0 \\ 1 \\ 0.8 \end{pmatrix}.$$

$$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{||v_1||^2} v_1 - \frac{\langle u_3, v_2 \rangle}{||v_1||^2} v_2 = \begin{pmatrix} -0.1923 \\ 0.3077 \\ 3 \\ -0.8077 \\ 0.1538 \end{pmatrix}.$$

So an orthogonal basis of W is

$$\left\{ \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1.6 \\ 0 \\ 1 \\ 0.8 \end{pmatrix}, \begin{pmatrix} -0.1923 \\ 0.3077 \\ 3 \\ -0.8077 \\ 0.1538 \end{pmatrix} \right\}.$$

Now, for the orthonormal basis, just divide by the length of each vector.

$$\left\{ \frac{1}{||v_1||} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \frac{1}{||v_2||} \begin{pmatrix} -1 \\ 1.6 \\ 0 \\ 1 \\ 0.8 \end{pmatrix}, \frac{1}{||v_3||} \begin{pmatrix} -0.1923 \\ 0.3077 \\ 3 \\ -0.8077 \\ 0.1538 \end{pmatrix} \right\}.$$