Math 6091/3091: Recitation 9

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Do **all** of the following problems.

(1) **(2 points)** Let $V = C[0,1]$ be the vector space consisting of continuous functions on the interval $[0,1]$. Define

$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt.$$

(a) Show that this function forms an inner product on V.

Sol:
Let $f, g, h \in V$ and $a \in \mathbb{R}$ then

$$\langle af+g, h\rangle = \int_0^1 (af(x)+g(x))h(x)dx = a\int_0^1 f(x)h(x)dx + \int_0^1 g(x)h(x)dx = a\langle f,h\rangle + \langle g,h\rangle$$

It is clear that

$$\langle f, g\rangle = \langle g, f\rangle$$

Finally

$$\langle f, f\rangle = \int_0^1 f^2(x)dx.$$

Now as $f(x)^2$ is a non negative function so its $\int_0^1 f^2(x)dx \geq 0$. Furthermore it is zero if and only if $f^2 = 0 \iff f = 0$. Therefore, $\langle \cdot, \cdot\rangle$ forms an inner product.

(b) Find the angle between the elements $f(x) = \sin x$ and $g(x) = \cos x$.

Sol:

$$\cos\theta = \frac{\int_0^1 \sin x\cos x dx}{\sqrt{\int_0^1 \sin^2 x dx}\cdot\sqrt{\int_0^1 \cos^2 x dx}} = \frac{\frac{1}{2}\sin 1}{\sqrt{\frac{1}{2}-\frac{1}{4}\sin 2}\cdot\sqrt{\frac{1}{2}+\frac{1}{4}\sin 2}}$$

(2) **(2 points)** Let $V = P_n(x)$ be the vector space consisting of polynomials of degree at most n with coefficients in $\mathbb{R}$. Fix $x_0, x_1, \cdots, x_n$ distinct real numbers. Define

$$\langle f(x), g(x)\rangle = f(x_0)g(x_0) + f(x_1)g(x_1) + \cdots + f(x_n)g(x_n).$$

(a) Show that this function forms an inner product on V.

Sol:
Let $f, g, h \in V$ and $\alpha \in \mathbb{R}$. Now,

$$\begin{aligned}\langle \alpha f+g, h\rangle &= (\alpha f(x_0)+g(x_0))h(x_0) + \cdots + (\alpha f(x_n)+g(x_n))h(x_n)\\ &= \alpha f(x_0)h(x_0) + \cdots + \alpha f(x_n)h(x_n) + g(x_0)h(x_0) + \cdots + g(x_n)h(x_n)\\ &= \langle \alpha f, h\rangle + \langle g, h\rangle.\end{aligned}$$

It is clear that $\langle f, g\rangle$. Finally,

$$\langle f, f\rangle = f(x_0)^2 + \cdots + f(x_n)^2 \geq 0.$$

Now if $\langle f, f\rangle = 0$, implies $f(x_0)^2 + \cdots + f(x_n)^2 = 0$. This is only true if and only if $f(x_i) = 0$ for all $i = 1, \cdots, n$. But then as f is a degree n polynomial, it has at most n zeros. But then this implies f has $n+1$ zeros. This forces $f = 0$.

(b) Let $V = P_2(x)$ and take $x_0 = -2, x_1 = 0$ and $x_2 = 2$. Compute $\langle f, g\rangle$ where $f = 1 + x^2$ and $g = 1 - x$. Find the angle between the above f and g.

Sol:

$$\langle f, g\rangle = f(x_0)g(x_0) + f(x_1)g(x_1) + f(x_2)g(x_2) = 5 \cdot 3 + 1 \cdot 1 + 5 \cdot (-1) = 11.$$
$$\langle f, f\rangle = 5^2 + 1^2 + 5^2 = 51$$
$$\langle g, g\rangle = 3^2 + 1^2 + (-1)^2 = 11$$

So,

$$\theta = \cos^{-1}\langle \frac{11}{\sqrt{51}\sqrt{11}}\rangle$$

(3) **(2 points)** Let W be a subspace in R^4 spanned by $w_1 = (1, 2, 1)$ and $w_2 = (1, -1, -1)$.

(a) Find the space $W^\perp$.

Sol:

$$W^\perp = \left\langle \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \right\rangle$$

(b) Let $u = (1, 0, 0)$. Write u in terms of $proj_W(u) + proj_{W^\perp}(u)$.

Sol:

$$proj_{W^\perp}(u) = \frac{\langle (1,0,0), (1,-2,3)}{||(1,-2,3)||^2}(1,-2,3) = \frac{1}{14}(1,-2,3).$$

Now,

$$proj_W(u) = u - proj_{W^\perp}(u) = \frac{1}{14}(-13, 2-3) \in W.$$

So,

$$u = \frac{1}{14}(-13, 2-3) + \frac{1}{14}(1,-2,3)$$

(4) **(3 points)** Let W_1 and W_2 be subspaces of $\mathbb{R}^n$.

(a) Show that $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$

Sol:
$x \in (W_1 + W_2)^\perp$ if and only if $\langle x, w_1\rangle = 0$ and $\langle x, w_2\rangle = 0$ for all $w_1 \in W_1$ and $w_2 \in W_2$. This happens if and only if $x \in W_1^\perp$ and $x \in W_2^\perp$ which is equivalent to $x \in W_1^\perp \cap W_2^\perp$.

(b) Show that $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$

Sol:
We assume the result of part (c). Then this follows from part (a).

$$\begin{aligned}(W_1 \cap W_2)^\perp &= ((W_1^\perp + W_2^\perp)^\perp)^\perp \\ &= W_1^\perp + W_2^\perp.\end{aligned}$$

(c) Show that $(W_1^{\perp})^{\perp} = W_1$.

Sol:
As $W^{\perp}$ is the orthogonal complement of W and $\dim W + \dim W^{\perp} = \dim V$ i.e. $V = W + W^{\perp}$. Therefore, the orthogonal complement of $W^{\perp}$, denoted by $(W_1^{\perp})^{\perp}$ must exactly be W. So,

$$(W_1^{\perp})^{\perp} = W.$$